Econ Theory Bull
DOI 10.1007/s40505-014-0042-8

RESEARCH ARTICLE

Proximate preferences and almost full revelation in the Crawford–Sobel game

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Received: 14 April 2014 / Accepted: 25 April 2014

Abstract Crawford and Sobel (Econometrica 50(6):1431–1451, 1982) is a seminal contribution that introduced the study of costless signalling of privately held information by an expert to a decision maker. Among the chief reasons for its widespread application is the comparative statics they develop between the extent of strategically transmitted information and the degree of conflict in the two players' preferences. This paper completes their analysis by establishing that in their general model, almost full revelation obtains as the two players' preferences get arbitrarily close to each other.

Keywords Cheap talk · Strategic information transmission · Bias · Convergence · Full revelation

JEL Classification C72 · D82 · D83

1 Introduction

Crawford and Sobel (1982) (CS) introduced a fairly general model of costless communication (cheap talk) by an informed expert to a decision maker. In particular, they

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Published online: 14 May 2014

study how the differing objectives of the decision maker from those of the expert cause the latter to strategically limit the transmission of information. The degree of conflict in the two players' preferences is parameterized by a scalar, say b, often referred to as the 'bias'. $b = 0$ denotes the case of no conflict at which the two players' ex-post optimal actions coincide. When there is no conflict, there can of course be a fully revealing equilibrium, but full revelation cannot occur if there is any conflict. In fact, they show that any equilibrium partitions the continuum of states into finitely many sets. In equilibrium, only the element of the partition in which the true state lies is reported. The CS model or its variants have been used extensively to study a wide range of issues: merits of open vs. closed legislative rules; the politics of special interest groups; doctor patient interactions; issues in corporate governance; and financial advice among many others.[1] It has also been an important tool in Organizational Economics to study the merits of delegating authority. (See Agastya et al. (2014) and the references therein.)

Among a number of comparative statics results that CS establish, one is on how the quality of information varies with a change in the bias. In particular, imposing a certain Condition (M), they show that the maximal possible number of elements in an equilibrium partition is non-decreasing. This weak monotonicity result leaves open the question of whether information is almost fully revealed as the two players' preferences tend to coincide, or if there is a discontinuity. This paper answers this question by showing that information is almost perfectly revealed (in the most informative equilibrium) by the expert if the degree of conflict is small enough. In fact, the result is obtained under milder conditions on the primitives used in CS to ensure Condition (M).

Continuity of payoffs and information structure is of natural interest in any study that involves a small divergence of objectives between the decision maker and the expert.[2] Spector (2000) was the first to address the "continuity" of the quality of information transmission as preferences get close for a particular payoff structure. Spector's analysis however assumes non-standard payoff structures for the two agents, utility functions that rule out even the quadratic loss function type payoffs that are typical in all applications of CS. In this paper, we present precise conditions under which the convergence to full information will hold within the context of the original CS paper. Moreover, as will be evident from the analysis below, the proof of the general case considered here is non-trivial.

The rest of the paper is divided into two main sections and an Appendix. Section 2 describes the results of CS. Section 3 contains the main result, namely, Theorem 3 that establishes the possibility of almost full revelation of sender's information when the bias is sufficiently small. Theorem 3 requires an assumption on the nature of the equilibria at the limit, i.e., when $b = 0$. We show (see Claim 1) the sufficient conditions

[1] See Gilligan and Krehbiel (1989), Krishna and Morgan (2001), Morgan and Stocken (2003), Benabou and Laroque (1992), Harris and Raviv (2008) among many other applications of the model. A significant theoretically motivated literature on strategic information transmission also exists which adds additional elements to the basic CS-model such as multi-dimensional type uncertainty, partial verifiability of actions, multiple experts reporting on the state, multiple principals, etc. For example, see Ambrus and Takahashi (2008) and the references therein or the recent survey by Sobel (2013).

[2] For example, this plays a critical role in discussing the relative merits of authority and delegation in Agastya et al. (2014).

imposed by CS (see their Theorem 2) imply the sufficient condition of our Theorem 3. This completes our Claim of convergence for the original CS model as well. The Appendix contains the proof of Claim 1.

2 The Crawford–Sobel game

There are two players, **S**(ender) and **R**(eceiver). The former privately observes the realization, denoted by θ, of a random variable that is distributed according to a differentiable probability distribution function $F(\theta)$, with a positive density $f(\theta)$ on [0, 1]. She then sends a message from a given message space $\mathcal{M}$. **R** observes **S**'s message and chooses an action which ends the game. We shall refer to θ as the "type" of **S**.

The ensuing vNM utility of **R** and **S** from an action $\xi \in \mathbb{R}$ at state θ is, respectively, given by $U^r(\xi, \theta)$ and $U^s(\xi, \theta, b)$, where $b \in \mathbb{R}$ is an arbitrary parameter. We shall retain the following assumptions on the vNM utility functions found in CS: U^i is twice continuously differentiable, $U^i_1(\xi, \theta) = 0$ for some y and $U^i_{11} < 0$. The optimal actions of the two players under full information of θ, $x^r(\theta) = \operatorname{argmax} U^r(\xi, \theta)$ and $x^s(\theta, b) = \operatorname{argmax}_\xi U^s(\xi, \theta, b)$, are then well defined. It is also assumed that $U^i_{12} > 0$, a sorting condition, which ensures that $x^r(\cdot)$ and $x^s(\cdot, b)$ are increasing. Hence, there is no loss of generality in setting

$$x^r(\theta) \equiv \theta$$

by "re-scaling ξ" and reinterpreting U^i. The parameter b, hereafter referred to as the "bias", is meant to capture the degree to which the two players' interests diverge. This dependence will become clear later, for the moment it suffices to assume that $x^r(\theta) = x^s(\theta, 0)$ for all θ so that there is no conflict on the ex-post optimal actions when $b = 0$.

Equilibrium A pure strategy of **S** is any (measurable) function $\sigma_s : \Theta \longrightarrow \mathcal{M}$ and **R**'s strategy is a mapping $\sigma_r : \mathcal{M} \longrightarrow \mathbb{R}$. Without loss of generality, the analysis is restricted to pure strategies. The composition of a strategy of **S** with a strategy of **R** yields an *outcome function* $Y : \Theta \longrightarrow \mathbb{R}$ where

$$Y(\theta) = \sigma_r(\sigma_s(\theta)).$$

If a strategy profile (σ_s, σ_r) is played, the action $Y(\theta)$ is chosen in state θ.

Definition 1 (*Equilibrium*) An equilibrium consists of a strategy profile (σ_s, σ_r) such that

$$U^s(Y(\theta), \theta, b) \geq U^s(Y(\theta'), \theta, b) \quad \forall \theta, \theta' \in \Theta, \tag{1}$$

and for every $m \in R(\sigma_s)$, where $R(\sigma_s) \subseteq \mathcal{M}$ denotes the range of σ_s,

$$\sigma_r(m) \in \operatorname*{argmax}_\xi \int_{\theta' \in \sigma_s^{-1}(m)} U^r(\xi, \theta') f(\theta') \mathrm{d}\theta', \tag{2}$$

whenever $\sigma_s^{-1}(m)$ is of non-zero probability.

Condition (1) is the usual incentive compatibility requirement on **S**'s behavior. Condition (2) is the requirement that at every m that is reported along the equilibrium path, **R**'s choice is a best response given his updated Bayesian posterior.[3] $Y(\cdot)$ is said to be an EOF if it is the outcome function of some perfect Bayesian equilibrium strategy profile (σ_s, σ_r).

Also, assume that the message space $\mathcal{M}$ is sufficiently rich to allow an onto function from itself to Θ. This will ensure that the non-existence of a fully revealing equilibrium is not an artifact of the model. If $b = 0$, since the two players have coincident interests, it is immediate that a fully revealing equilibrium exists. CS offer a complete characterization of all the EOF with $b \neq 0$ as follows: First, define for any $a \le a'$,

$$x(a, a') = \underset{\xi}{\operatorname{argmax}} \int_{a}^{a'} U^r(\xi, \theta) f(\theta) \mathrm{d}\theta \tag{3}$$

to be the optimal action of **R** in the event he knows that θ lies in the interval $[a, a']$. Also for any $a \le \xi \le a'$, write

$$V(a, \xi, a', b) = U^s(x(a, \xi), \xi, b) - U^s(x(\xi, a'), \xi, b). \tag{4}$$

Next, let $\mathbf{a} = (a_0, a_1, \ldots, a_N)$ denote a typical partition of Θ into N sub-intervals where $0 = a_0 < a_1 < \cdots < a_N = 1$ are the end points of the sub-intervals.

Theorem 1 (Crawford and Sobel (1982)) *Suppose $x^r(\theta) \neq x^s(\theta, b)$ for all θ.*

1. *Y is an EOF if and only if there exists a partition $\mathbf{a} = (a_0, a_1, \ldots, a_N)$ of Θ such that*

$$Y(\theta) = x(a_i, a_{i+1}) \tag{5}$$
$$\text{and} \quad V(a_{i-1}, a_i, a_{i+1}, b) = 0 \tag{6}$$

   for all $\theta \in [a_{i-1}, a_i]$, and for $i = 1, \ldots, N-1$.
2. *There exists a finite integer N_b such that an equilibrium described in Part 1 exists if and only if $N \le N_b$.*

In words, when $b \neq 0$, every equilibrium involves partitioning the continuum of states Θ into some N sub-intervals. In what follows, we will refer to such an

[3] The analysis here is presented in terms of behavioral (pure) strategies whereas CS work with distributional strategies. This difference is inessential here. Furthermore, the definition of an equilibrium must specify players' beliefs at all information sets, including out of the equilibrium path, as well as (1) and (2). Insofar as our concern is only in the characterization of the equilibrium outcome function (EOF), this is without loss of generality because, given a strategy profile (σ_s, σ_r) such that (1) and (2) hold, pick $\hat{\theta}$ arbitrarily and let $\hat{m} = \sigma_s(\hat{\theta})$. For any $m \in \mathcal{M} \setminus R(\sigma_s)$, which represents an unreached node in the candidate equilibrium (σ_s, σ_r), prescribe the beliefs of **R** at m to be the same as those at $\hat{m}$ and redefine $\sigma_r(m) = \sigma_r(\hat{m})$. That is, **R** behaves at any unreached equilibrium message exactly as he does upon hearing $\hat{m}$. Since the original incentive compatibility conditions prevent any type (other than $\hat{\theta}$) from mimicking the behavior of $\hat{\theta}$, with the above prescribed beliefs, every type of **S** has an incentive to weakly report $\sigma_s(\theta)$ and makes (σ_s, σ_r) a perfect Bayesian equilibrium, in the sense of Fudenberg and Levine (1990).

equilibrium as an *N-equilibrium*. All types of **S** within a sub-interval pool to send the same message. The type at the edge of an interval, say a_i, should be indifferent between pooling with the types in $[a_{i-1}, a_i]$ and those in $[a_i, a_{i+1}]$, which gives (5). Equation (5) ensures that **R**'s action is a best response at each event that occurs with a positive probability along the equilibrium path. N_b, being a finite integer, places a bound on the extent of information transmission that occurs in any equilibrium. We shall refer to N_b-equilibrium as the *most informative equilibrium*.

Comparative statics Theorem 1 shows that the quality of strategically transmitted information is bounded away from full information when the interests do not coincide. For this, it is enough to assume that $x^s(\theta, b) \neq x^r(\theta)$ for all θ. Later in their paper, to study the comparative statics of the players' welfare and the quality of information transmitted as players' preferences get closer (in terms of b), CS assume that

$$U^r(y, \theta) \equiv U^s(y, \theta, 0) \quad \text{and} \quad U^s_{13}(y, \theta, b) > 0 \quad \text{if } b > 0.$$

The above inequality ensures that for each θ, an increase in the distance of b from 0 shifts the preferences of **S** from those of **R** for all θ. In particular, $x^s(\theta, b) \neq \theta$ for all θ when $b \neq 0$. Furthermore, they introduce a certain property which they refer to as Condition (M), which essentially amounts to assuming that for every $N \leq N_b$, the solution to the system of equations defined by (5) is unique for each N. They also develop a sufficient condition on the primitives for the satisfaction of Condition (M): first define $G(\xi, \theta, b) := U^s_1(\xi, \theta, b) + U^s_2(\xi, \theta, b)$ and $\hat{G}(\xi, \theta, b) := \int_0^\theta U^s_{11}(\xi, \theta', b)\mathrm{d}F(\theta') + f(\theta)U^s_1(\xi, \theta, b)$.

Condition (M_b) Given b, $G(\xi, \theta, b)$ is non-decreasing in ξ for all b and $\hat{G}(\xi, \theta, 0)$ is non-increasing in ξ.

Of the several comparative statics results that are established in CS, the one that is most relevant for our study may be formally stated as follows:

Theorem 2 (Crawford and Sobel (1982)) *Suppose Condition (M_b) holds for all b. Then Condition (M) holds, and consequently N_b is non-decreasing.*

We reiterate that even if one were to assume that $N_b \to \infty$ as $b \to 0$, it still does not follow that all information is fully revealed in the limit. For, it still leaves open the possibility that the limiting equilibrium partition has a non-degenerate interval. The next section shows that this cannot happen, not only under the conditions of Theorem 2, but also more generally.

3 Proximate preferences and informativeness of equilibria

We begin by noting that when $b = 0$ an N-equilibrium exists for all N. To see this, begin by considering

$$\Delta_N = \left\{(a_1, \ldots, a_{N-1}) \in \Theta^{N-1} : a_{i-1} \leq a_i, \quad i = 1, \ldots, N-1\right\}.$$

(Recall $a_0 := 0$ and $a_N = 1$.) Any $\boldsymbol{a} \in \Delta_{N-1}$ describes a way of partitioning Θ into some $k \leq N$ intervals. For any such $\boldsymbol{a}$, let

$$\Pi_N(\boldsymbol{a}) = \sum_{i=1}^{N} \int_{a_{i-1}}^{a_i} U^s(x(a_{i-1}, a_i), \theta, 0) \mathrm{d}F(\theta)$$

and

$$\boldsymbol{a}_N^* \in \underset{\boldsymbol{a}}{\operatorname{argmax}} \, \Pi_N(\boldsymbol{a}),$$

which is of course well defined, since Δ_N is compact and Π_N is continuous.

Later (see Lemma 3), we will show that $a_{i-1,N}^* < a_{i,N}^*$. Therefore, using the Envelope Theorem, the first-order conditions for characterizing $\boldsymbol{a}_N^*$ are given by

$$\frac{\partial \Pi_N\left(\boldsymbol{a}_N^*\right)}{\partial a_i} = V\left(a_{N,i-1}^*, a_{N,i}^*, a_{N,i+1}^*, 0\right) f(a_i^*) = 0, \quad i = 1, \ldots, N-1.$$

Comparing the above first-order conditions with the conditions given in Theorem 1 shows that $\left(\boldsymbol{x}_N^*, \boldsymbol{a}_N^*\right)$, where $\boldsymbol{x}_N^* = (x_1(a_{N,0}^*, a_{N,1}^*), \ldots, x_i(a_{N,i-1}^*, a_{N,i}^*), \ldots, x_N (a_{N,N-1}^*, a_{N,N}^*))$, describes an N-equilibrium outcome and $\boldsymbol{a}_N^*$ is an N-equilibrium partition of the CS game *when preferences of* **R** *and* **S** *coincide*, i.e., for $b = 0$.

Definition 2 An N-equilibrium partition of the CS-game with zero bias $\boldsymbol{a}_N^*$ is said to be *regular* if the Hessian $H(\Pi_N)\left(\boldsymbol{a}_N^*\right)$ is invertible.

Let $\Pi_N(b)$ denote the equilibrium payoff of **R** in an N-equilibrium when the bias is b. Recall that the norm of a partition $\boldsymbol{a}$, denoted by $\|\boldsymbol{a}\|$, is the length of its largest element. The full information payoff of **R** is

$$\Pi := E[U^s(\theta, \theta, 0)].$$

Our main result is as follows:

Theorem 3 (Main result) *Assume that $\boldsymbol{a}_N^*$ is regular for all N sufficiently large. For any $\varepsilon > 0$, there exists $\delta > 0$, such that whenever $|b| < \delta$, there exists an N-equilibrium such that*

1. $|\Pi_N(b) - \Pi| < \varepsilon$ *and*
2. $\|a_N\| < \varepsilon$.

Before beginning to prove the theorem, we briefly comment on the requirement of regularity. It is well known that in general, a slight perturbation in the payoffs of a game can result in a large change to the equilibrium, or even cause non-existence. In the CS-game, an N-equilibrium exists when $b = 0$ for all integers N. With the introduction of a bias $b \approx 0$ it is apriori possible that no equilibrium exists for values of N beyond some integer. Regularity ensures that for every N, there is a N-equilibrium for some b close to zero.[4] The following Claim gives conditions on primitives that ensure regularity.

[4] Equilibrium is after all obtained as the solution to a fixed-point problem. Therefore, the question of the existence of such an equilibrium reduces to asking for the continuity of a fixed-point mapping. Regularity is

Claim 1 *Suppose Condition (M_b) holds when $b = 0$. $\boldsymbol{a}_N^*$ is regular for all N.*

Proof of the above Claim 1 is in the Appendix. Claim 1, together with Theorems 2 and 3, implies convergence to full revelation at least for the class of environments presented in Theorem 2 of CS.

3.1 Proof of Theorem 3

The two key steps required for the proof of the theorem may be stated as Lemmas 1 and 2. Lemma 1 concerns the behavior of the N-equilibrium when the players' preferences are identical, i.e., the case when the bias $b = 0$. It shows in particular that in this case information converges to full information and the payoffs are the full information payoffs as $N \to \infty$.

Lemma 1 $\lim_{N\to\infty} \left\|\boldsymbol{a}_N^*\right\| = 0$ *and* $\lim_{N\to\infty} \Pi_N = \Pi$.

Lemma 2 considers the existence of an N-equilibrium for an arbitrarily chosen N. It shows that for a small enough bias ($b \approx 0$), such an equilibrium exists.

Lemma 2 *Suppose $\boldsymbol{a}_N^*$ is regular for some N. There exists $\delta_N > 0$ and a continuous function $\varphi : (-\delta_N, \delta_N) \longrightarrow \dot{\Delta}_N$ such that (i) $\varphi(0) = a_N^*$ and (ii) $\varphi(b)$ is an N-equilibrium partition when the bias is b.*

With the above two Lemmas in hand, choose any $\varepsilon > 0$. Using Lemma 1, choose an integer N_ε such that $\left\|\boldsymbol{a}_N^*\right\| < \varepsilon/2$ for all $N \geq N_\varepsilon$. Pick δ_{N_ε} as per Lemma 2 and consider the function φ be as given there. There must exist a $\delta' \leq \delta_{N_\varepsilon}$ such that $\left|\Pi_N(\varphi(b)) - \Pi_N\left(\boldsymbol{a}_N^*\right)\right| < \varepsilon$, since Π_N and φ are continuous and $\varphi(0) = \boldsymbol{a}_N^*$. Similarly, there must exist a $\delta'' \leq \delta_N$ such that $\left|\|\varphi(b)\| - \left\|\boldsymbol{a}_N^*\right\|\right| < \varepsilon/2$ whenever $|b| < \delta''$ and for all such b, $\|\varphi(b)\| < \varepsilon$. The statement of the theorem holds for $\delta = \min\{\delta', \delta'', \delta_N\}$. Proof of the theorem is then complete upon proving Lemmas 1 and 2.

Proofs of the above Lemmas depend on establishing that **R**'s payoff-maximizing partition $\boldsymbol{a}_N^*$ in Δ_N does in fact partition Θ into *exactly* N sub-intervals. This will be an easy consequence of Lemma 3 below.

Given $a < a'$ and $\xi \in [a, a']$, define

$$W(a, \xi, a') = \int_a^\xi U^s(x(a, \xi), \theta, 0)\mathrm{d}F(\theta) + \int_\xi^{a'} U^s(x(\xi, a'), \theta, 0)\mathrm{d}F(\theta).$$

and

$$z(a, a') = \operatorname*{argmax}_{\xi\in[a,a']} W(a, \xi, a').$$

Footnote 4 continued
very close in spirit to the type of conditions that are imposed to ensure the continuity of fixed-point mappings with respect to some parameter. McLennan (2012) surveys the fixed-point theory from a perspective useful for Economics.

Lemma 3 *$a < z(a, a') < a'$ for any $a < a'$. Moreover,*

$$V(a, z(a, a'), a) = 0. \tag{7}$$

Proof (Lemma 3) A routine application of the Envelope Theorem gives us $W_2(a, \xi, a') = V(a, \xi, a')$. Recalling that θ uniquely maximizes $U^s(\cdot, \theta, 0)$ for all θ, $W_2(a, a, a') = V(a, a, a') > 0$ and $W_2(a, a', a') = V(a, a', a') < 0$. Therefore $z(a, a')$ must lie in (a, a'). Equation 7) is merely the first-order necessary condition for an interior maximum.

Corollary 1 *For each N, $\boldsymbol{a}_N^* = (a_{N,1}^*, \ldots, a_{N,N-1}^*)$ satisfies $a_{N,i-1}^* < a_{N,i}^*$ for $i = 1, \ldots, N$ and hence $\nabla \Pi_N(\boldsymbol{a}^*) = 0$.*

In what follows, we will write a_i^* to denote $a_{N,i}^*$, and the context makes it clear what is the particular underlying value of N. We do not anticipate any confusion.

Proof (Lemma 1) Let $\Pi_n \equiv \Pi_n(\boldsymbol{a}_n^*)$. Note that (Π_n) is a non-decreasing sequence, since $\Delta_n \subset \Delta_{n+1}$ for all n and it is clearly bounded above by Π. Therefore, it must be the case that

$$\lim_{n\to\infty} (\Pi_{n+1} - \Pi_n) = 0. \tag{8}$$

It suffices to show that $\lim_{n\to\infty} \|\boldsymbol{a}_n^*\| = 0$, as the other Claim of the Lemma is an immediate corollary to this fact. Assume, by way of contradiction, that $\lim_{n\to\infty} \|\boldsymbol{a}_n^*\| \neq 0$. We shall show that this results in a contradiction to (8).

For each n, let $(a_{n_k}^*, a_{n_k+1}^*)$ denote the largest interval (by length) in the partition $\boldsymbol{a}_n^*$. Assume, without loss of generality (otherwise take suitable subsequences), that $a_{n_k}^* \to \alpha$ and $a_{n_k+1}^* \to \beta$. If $\lim_{n\to\infty} \|a_n^*\| \neq 0$, then $\alpha < \beta$. By the maximum theorem, $z(\cdot)$ is continuous and therefore $z(a_N^*, a_{N+1}^*) \to z(\alpha, \beta)$. Let $\widehat{\boldsymbol{a}}$ denote the partition with $n+1$ intervals obtained from $\boldsymbol{a}_n^*$ by splitting $[a_{n_k}^*, a_{n_k+1}^*]$ into $[a_{n_k}^*, \xi^*]$ and $[\xi^*, a_{n_k+1}^*]$, where $\xi^* = z(a_{n_k}^*, a_{n_k+1}^*)$. Also, let $\widehat{\boldsymbol{x}} \in \Delta_{n+1}$ that is obtained from $\boldsymbol{x}_n^*(b)$ by choosing $y(a_{n_k}^*, \xi^*, 0)$ and $y(\xi^*, a_{n_k+1}^*, 0)$ in the newly introduced intervals. Note that

$$\Pi_{n+1}(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{a}}, b) = \Pi_n + W\left(a_n^*, z\left(a_{n_k}^*, a_{n_k+1}^*\right), a_{n_k+1}^*\right) - W\left(a_{n_k}^*, a_{n_k}^*, a_{n_k+1}^*\right).$$

Since $\Pi_{n+1}(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{a}}, b) \leq \Pi_{n+1}$,

$$\Pi_{n+1} - \Pi_n \geq W\left(a_n^*, z\left(a_{n_k}^*, a_{n_k+1}^*\right), a_{n_k+1}^*\right) - W\left(a_{n_k}^*, a_{n_k}^*, a_{n_k+1}^*\right)$$

for all n. Now, the RHS converges to $\delta = W(\alpha, z(\alpha, \beta), \beta) - W(\alpha, \alpha, \beta)$ and by Lemma 3, $\delta > 0$. Thus, the sequence $(\Pi_{n+1} - \Pi_n)$ cannot converge to zero, i.e., (8) is violated.

Proof (Lemma 2) Let $\dot{\Delta}_N$ denote the interior of Δ_N and define $g : \dot{\Delta}_N \times \mathbb{R} \longrightarrow \mathbb{R}^{N-1}$ where $g(\boldsymbol{a}, b) = (g_1(\boldsymbol{a}, b), \ldots, g_{N-1}(\boldsymbol{a}, b))$ is defined by

$$g_i(\boldsymbol{a}, b) = V(a_{i-1}, a_i, a_{i+1}, b) f(a_i).$$

Note that $\boldsymbol{a} \in \Delta_N$ is an N-equilibrium of the CS game with bias b if and only if $g(\boldsymbol{a}, b) = 0$. Thus, the proof is complete, if for all b sufficiently small, we exhibit a partition $\boldsymbol{a}^b \in \dot{\Delta}_N$ such that $g\left(\boldsymbol{a}^b, b\right) = 0$.

Note that $g\left(\boldsymbol{a}_N^*, 0\right) = \nabla\Pi\left(\boldsymbol{a}_N^*\right) = 0$. Moreover, the Jacobian $J_{\boldsymbol{a}}g\left(\boldsymbol{a}_N^*, 0\right) = \mathrm{H}(\Pi_N)\left(\boldsymbol{a}_N^*\right)$ which, given our hypothesis that $\boldsymbol{a}_N^*$ is regular, is invertible. By the implicit function theorem, there exists a $\delta_N > 0$ and a continuous function $\varphi : (-\delta_N, \delta_N) \longrightarrow \dot{\Delta}_N$ such that $g(\varphi(b), b) = 0$ for all $b \in (-\delta_N, \delta_N)$ and $\varphi(0) = \boldsymbol{a}_N^*$. For all such b, $\varphi(b)$ is an N-equilibrium partition of the CS-game when the bias is b.

3.2 Quadratic loss, Spector (2000) and Theorem 3

We have, thus, established the continuity of the equilibrium payoffs in the CS game as preferences of the sender converge to those of receiver. Interestingly, the result obtains under assumptions at least as general as those found in the original paper, Crawford and Sobel (1982). Spector (2000) considers the special case where

$$U^s(\xi, \theta, b) = u(\xi, \theta) + bv(\xi, \theta)$$

for a pair of functions u and v that satisfy the usual assumptions so that U^s will satisfy the usual assumptions. However, the typical parametric form in virtually all the applications of CS model in the literature use the quadratic form

$$U^s(\xi, \theta, b) = -(\xi - b - \theta)^2.$$

Clearly, this specification cannot be embedded within Spector's framework. Yet, it may be readily verified that

$$G(\xi, \theta, 0) = 0, \quad \text{and} \quad \hat{G}(\xi, \theta, 0) = -2F(\theta) - f(\theta)(\xi - \theta).$$

Claim 1 and hence Theorem 3 readily show that convergence to full information occurs with quadratic loss functions.

Appendix

Proof (Claim 1) Throughout $U(\xi, \theta) \equiv U^s(\xi, \theta, 0)$. We will first show that for $a < a'$,[5]

$$x_1(a, a') + x_2(a, a') \leq 1. \tag{9}$$

From the first-order condition that determines $x(a, a')$ we have

$$\int_a^{a'} U_1(x(a, a'), \theta)\mathrm{d}F(\theta) \equiv 0.$$

[5] Here, and elsewhere in the proof when $a = 0$ or $a' = 1$, $x_1(a, a')$ and $x_2(a, a')$ should be interpreted as the right and left derivatives respectively. Similarly for $V_1(a, \hat{a}, a')$ and $V_3(a, \hat{a}, a')$ later on in the proof.

Setting $\xi := x(a, a')$ and partially differentiating the above with respect to a and a' gives:

$$x_1(a, a') \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) + f(a')U_1(\xi, a') = 0$$

$$x_2(a, a') \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) - f(a)U_1(\xi, a) = 0.$$

Summing the above, we have

$$\begin{aligned}
0 &= \left(x_1(a, a') + x_2(a', a')\right) \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) \\
&\quad + f(a')U_1(\xi, a') - f(a)U_1(\xi, a) \\
&= \left(x_1(a, a') + x_2(a', a')\right) \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) \\
&\quad + \hat{G}(\xi, a') - \hat{G}(\xi, a) - \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) \\
&= \left(x_1(a, a') + x_2(a', a') - 1\right) \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta) + \hat{G}(\xi, a', 0) - \hat{G}(\xi, a, 0) \\
&\leq \left(x_1(a, a') + x_2(a', a') - 1\right) \int_a^{a'} U_{11}(\xi, \theta)\mathrm{d}F(\theta),
\end{aligned}$$

where the inequality is from the hypothesis of the Claim, that $\hat{G}(\xi, a, 0)$ is non-increasing in a. The fact that $U_{11} < 0$ yields (9). Write

$$V^i(\boldsymbol{a}) := V\left(a^*_{i-1}, a^*_i, a^*_{i+1}, 0\right) f\left(a^*_i\right) = \left(U^r_1\left(x^i, a_i\right) - U^r_1\left(x^{i+1}, a_i\right)\right) f\left(a^*_i\right)$$

where $x^i := x(a^*_{i-1}, a^*_i)$ and for convenience, set $n = N - 1$. Then,

$$H(\Pi_N)(\boldsymbol{a}^*_N) = \begin{pmatrix}
\alpha_1 & \gamma_1 & & \cdots & 0 \\
\beta_1 & \alpha_2 & \ddots & & \vdots \\
 & \beta_2 & \ddots & \gamma_{n-2} & 0 \\
\vdots & & \ddots & \alpha_{n-1} & \gamma_{n-1} \\
0 & & \cdots & \beta_{n-1} & \alpha_n
\end{pmatrix}$$

where $\alpha_i = \dfrac{\partial V^i(\boldsymbol{a}_N^*)}{\partial a_i}$ for $i = 1, \ldots, n$, $\beta_i = \frac{\partial V^i(\boldsymbol{a}_N^*)}{\partial a_{i+1}}$ and $\gamma_i = \frac{\partial V^{i+1}(\boldsymbol{a}_N^*)}{\partial a_i}$ for $i = 1, \ldots, n-1$. Of course, by symmetry of the Hessian, we have $\beta_i = \gamma_i$. Therefore, for any vector $\boldsymbol{y} = (\xi_1, \ldots, \xi_n) \neq \boldsymbol{0}$, setting $\beta_0 = V_1(0, a_1^*, a_2^*) f(a_1)$ and $\beta_N = V_3(a_{n-1}^*, a_n^*, 1) f(a_n)$ we have

$$\boldsymbol{y}^t H\left(\boldsymbol{a}_N^*\right) \boldsymbol{y} = \sum_{i=1}^{n} \alpha_i \xi_i^2 + 2 \sum_{i=1}^{n-1} \beta_i \xi_i \xi_{i+1}.$$

Note that $\beta_i > 0$ (see Lemma 2 in CS for instance). Assume, for the moment, that

$$\alpha_i \leq -(\beta_{i-1} + \beta_i). \tag{10}$$

Hence,

$$\begin{aligned} \boldsymbol{y}^t H\left(\boldsymbol{a}_N^*\right) \boldsymbol{y} &\leq -\sum_{i=1}^{n-1} \beta_i \left(\xi_i^2 + \xi_{i+1}^2 - 2\xi_i \xi_{i+1}\right) - \beta_0 \xi_1^2 - \beta_n \xi_n^2 \\ &= -\sum_{i=1}^{n-1} \beta_i (\xi_i - \xi_{i+1})^2 - \beta_0 \xi_1^2 - \beta_n \xi_n^2 < 0. \end{aligned}$$

In other words, $H(\boldsymbol{a}_N^*)$ is negative definite, and hence invertible. So, it remains to show (10) to complete the proof. Using the notation $x^i = x(a_{i-1}^*, a_i^*)$ and $x_j^i = x_j(a_{i-1}^*, a_i^*)$, for $j = 1, 2$,

$$\begin{aligned} \frac{1}{f(a_i^*)} \times \frac{\partial V^i}{\partial a_{i-1}} &= x_1^i U_1\left(x^i, a_i^*\right) \\ \frac{1}{f(a_i^*)} \times \frac{\partial V^i}{\partial a_{i+1}} &= -x_2^{i+1} U_1\left(x^{i+1}, a_i^*\right) \\ \frac{1}{f(a_i^*)} \times \frac{\partial V^i}{\partial a_i} &= x_2^i U_1\left(x^i, a_i^*\right) + U_2\left(x^i, a_i^*\right) \\ &\quad - x_1^{i+1} U_1\left(x^{i+1}, a_i^*\right) - U_2\left(x^{i+1}, a_i^*\right). \end{aligned}$$

Therefore, for $i = 1, \ldots, n$,

$$\begin{aligned} \frac{1}{f(a_i^*)} \times \left(\frac{\partial V^i}{\partial a_{i-1}} + \frac{\partial V^i}{\partial a_i} + \frac{\partial V^i}{\partial a_{i+1}}\right) &= \left(x_1^i + x_2^i\right) U_1\left(x^i, a_i^*\right) + U_2\left(x^i, a_i^*\right) \\ &\quad - \left(x_1^{i+1} + x_1^{i+1}\right) U_1\left(x^{i+1}, a_i^*\right) \\ &\quad - U_2\left(x^{i+1}, a_i^*\right). \end{aligned}$$

Since $x^i < a_i^* < x^{i+1}$, for all i, $U_1(x^i, a_i^*) > 0 > U_1(x^{i+1}, a_i^*)$. Using (9), we then have

$$\frac{\partial V^i}{\partial a_{i-1}} + \frac{\partial V^i}{\partial a_i} + \frac{\partial V^i}{\partial a_{i+1}} \leq f(a_i^*) \times \left(G\left(x^i, a_i^*, 0\right) - G\left(x^{i+1}, a_i^*, 0\right)\right) \leq 0,$$

which in turn completes the proof of (10) for $i = 1, \ldots, n$.

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