

# Supplementary Appendix for Tacit Collusion with Price-Matching Punishments

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In this Supplementary Appendix we provide the analysis of three extensions to our paper “Tacit Collusion with Price-Matching Punishments”. Appendix A provides an analysis of tacit collusion with price-matching punishments in the Hotelling model of spatial competition. Appendix B provides the full details of how to get the critical discount factor under Nash reversion for any collusive price and comparing it to the counterpart under price-matching punishments in the linear demand example considered in Section 3. Appendix C considers the extension of the analysis to  $T$ -period punishments instead of using infinite punishments.

## 1 Appendix A: Hotelling Model

We use the standard Hotelling model of differentiated products. There are two firms, each producing a single product at constant marginal cost, which is normalized to zero without loss of generality. The two firms are located at either end of a linear city of length one. Consumers are distributed uniformly along the linear city and their density is normalized to one. Consumers will buy the product with the lowest total price, where the total price includes the transportation cost in the case the consumer is located some distance from the firm. The constant per unit (of distance) transport cost,  $\tau$ , serves as a measure of product differentiation. Consumers’ reservation price is  $v$ ; consumers will buy at most one unit of product as long as the total price is no more than  $v$ , otherwise they will not purchase. We will assume conditions such that at the collusive price, all consumers do

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buy the product (in equilibrium).

When the prices of the two firms are sufficiently close, both firms will have positive demands, and we can easily get firm  $i$ 's demand function in terms of  $p_i$  and  $p_j$ . However, when the price of the two firms strongly diverge, the high price firm will receive no demand, while the low price firm captures the entire market. Specifically, for any price  $p_j$ , when  $0 < p_i \leq p_j - \tau$ , firm  $i$  captures the entire market; while its demand becomes zero when  $p_i \geq p_j + \tau$ . Thus, firm  $i$ 's demand function is

$$q_i = \frac{1}{2} + \frac{p_j - p_i}{2\tau} \quad \text{if} \quad p_j - \tau < p_i < p_j + \tau \quad (1)$$

$$= 1 \quad \text{if} \quad 0 < p_i \leq p_j - \tau \quad (2)$$

$$= 0 \quad \text{if} \quad p_i \geq p_j + \tau. \quad (3)$$

It is straightforward to check that the monopoly prices, quantities, and profits are  $p^m = v - \tau/2$ ,  $q^m = 1/2$  and  $\pi^m = (v - \tau/2)/2$  respectively. Similarly, the one-shot Nash equilibrium prices, quantities, and profits are  $p^n = \tau$ ,  $q^n = 1/2$ , and  $\pi^n = \tau/2$  respectively. The properties we assumed in Section 2 will then apply given profits are smooth over the relevant range of prices except that  $\pi(p)$  is not strictly concave nor differentiable when  $p = p^m$ .<sup>1</sup> An implication of the Hotelling model with unit demands is that profit functions are not flat at the monopoly price. Rather, the monopoly price is just determined by the condition that consumers do not drop out. This means a small decrease in prices from the monopoly price will have a first order impact on collusive profits. As a result, we find the monopoly prices is sustainable under price-matching punishments when  $\delta \geq 1 - \tau/p^m$  in the special case of the Hotelling model. As a result, Proposition 2 does not hold.

We first investigate whether collusion is sustainable under price-matching punishments. Firm  $i$ 's best response to any price  $p_j$  set by firm  $j$  is as follows:

$$p_i = \frac{p_j + \tau}{2} \quad \text{if} \quad 0 \leq p_j < 3\tau \quad (4)$$

$$= p_j - \tau \quad \text{if} \quad 3\tau \leq p_j < v + \tau/2 \quad (5)$$

$$= v - \tau/2 \quad \text{if} \quad p_j \geq v + \tau/2. \quad (6)$$

If the rival's price is low, we get the standard (linear) upward sloping best response functions. If the rival's price is high, firm  $i$  will set a relatively low price and capture the

<sup>1</sup>Specifically, note that  $\partial^2 \pi_i / \partial p_i^2 = -1/\tau < 0$  and  $\partial^2 \pi_i / \partial p_i \partial p_j = 1/(2\tau) > 0$ , so that (1) and (2) in the text hold.  $\pi(p) = p/2$  when  $p \leq p^m$  and  $\pi(p) = p(v - p)/\tau$  when  $p^m < p \leq v$  and it follows that  $d^2 \pi / dp^2 = 0$  when  $p < p^m$  and that  $\pi(p)$  is not differentiable at  $p = p^m$  so that (3) in the text does not hold.

whole market, while it can do the same thing by setting the monopoly price if the rival sets a very high price. The linear best response function (4) applies for any  $p_j$  between  $p^n$  and  $p^m$  when  $3\tau/2 < v \leq 7\tau/2$ , while for higher  $v$ , whether firm  $i$ 's best response is determined by (4) or (5) depends on whether  $p_j$  is closer to  $p^n$  or  $p^m$ .

Now we consider whether a firm, say firm 1, has an incentive to deviate from some collusive price  $p^n < p^c \leq p^m$  given that both firms adopt price-matching punishment strategies. When we consider firm 1's profit if it defects, we need to distinguish three cases: (1)  $3\tau/2 < v \leq 7\tau/2$ ; (2)  $v > 7\tau/2$  and  $p^n < p^c \leq 3\tau$ ; and (3)  $v > 7\tau/2$  and  $3\tau < p^c \leq p^m$ . In the first and second cases, firm 1's best response is given by (4), and its optimal deviation price will exceed  $(p^c + \tau)/2$ . In the third case, firm 1's best response is given by (5), and its optimal deviation price will be greater than or equal to  $p^c - \tau$ . Firm 1's profit if it deviates will be

$$p_1 \left( \frac{1}{2} + \frac{p^c - p_1}{2\tau} \right) + \frac{\delta}{1 - \delta} \frac{p_1}{2}, \quad (7)$$

where the first term is firm 1's current (one period) profit and the second term is the present value of its profit in subsequent periods given price-matching punishments. This function also gives the present discounted value of firm 1's profits if it continues to collude, in which case  $p_1 = p^c$ . We also notice that  $(p^c + \tau)/2 \geq p^c - \tau$  when  $v > 7\tau/2$  and  $p^n < p^c \leq 3\tau$  and that  $p^c - \tau > (p^c + \tau)/2$  when  $v > 7\tau/2$  and  $3\tau < p^c \leq p^m$ . Therefore, firm 1 will choose  $p_1$  to maximize (7) subject to

$$\max((p^c + \tau)/2, p^c - \tau) \leq p_1 \leq p^c.$$

Solving this constrained maximization problem, we obtain firm 1's optimal deviation price, which is less than the collusive price when  $p_1 = p^c$  if  $p^c > \tau/(1 - \delta)$  and equals the collusive price if  $p^c \leq \min\{\tau/(1 - \delta), p^m\}$ . Thus, under price-matching punishments, collusion can be sustained for any collusive price satisfying

$$p^n < p^c \leq \min\{\tau/(1 - \delta), p^m\}, \quad (8)$$

but cannot be sustained for any higher collusive price.

The critical discount factor necessary to sustain collusion is defined by  $\delta \geq \delta^c = 1 - \tau/p^c$ , which increases as  $\tau$  decreases. So in the Hotelling model, we also find that increased product differentiation makes collusion easier to sustain. Finally, Proposition 5 in the text can be examined for this example by comparing the sustainable region of

collusive prices under price-matching punishments with Nash-reversion. Given the existing literature only examines collusion under Nash-reversion in the Hotelling model with respect to the monopoly price, we again need to distinguish three cases: (1)  $3\tau/2 < v \leq 7\tau/2$ ; (2)  $v > 7\tau/2$  and  $p^n < p^c \leq 3\tau$ ; and (3)  $v > 7\tau/2$  and  $3\tau < p^c \leq p^m$ . In the first and second case, the defector's optimal deviation price is  $(p^c + \tau)/2$  and thus the deviation profit is  $\pi_i^r = (p^c + \tau)^2/(8\tau)$ ; in the last case, its optimal deviation price is  $p^c - \tau$  and the deviation profit is  $\pi_i^r = p^c - \tau$ .

The collusive profit is  $\pi_i^c = p^c/2$  and the Nash equilibrium profit is  $\pi_i^n = \tau/2$ . Substituting  $\pi_i^r$ ,  $\pi_i^c$  and  $\pi_i^n$  into the formula of critical discount factor under Nash-reversion  $\delta^c = (\pi_i^r - \pi_i^c) / (\pi_i^r - \pi_i^n)$ , we obtain the critical discount factor under Nash-reversion for each case. In the first and second case,

$$\delta^c = \frac{p^c - \tau}{p^c + 3\tau}. \quad (9)$$

In the last case,

$$\delta^c = \frac{p^c - 2\tau}{2p^c - 3\tau}. \quad (10)$$

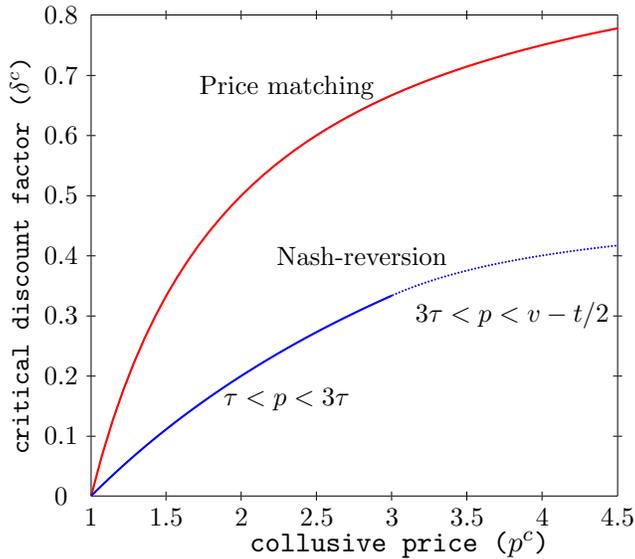


Figure 1: ( $v = 5$  and  $\tau = 1$ ).

It can be shown that for the Hotelling model, the set of discount factors and collusive prices sustainable under price-matching punishments is a proper subset of those sustainable

under Nash-reversion, as suggested by Proposition 5. Figure 1 illustrates the relationship between  $\delta^c$  and  $p^c$  for  $v = 5$  and  $\tau = 1$ , comparing it to the relationship between  $\delta^c$  and  $p^c$  under Nash-reversion.

## 2 Appendix B: The critical discount factor under Nash reversion in the linear demand example

Here we show how to obtain the critical discount factor under Nash reversion for any collusive price in the linear demand example of Section 3. To obtain the critical discount factor, we need to distinguish three cases: (1)  $0 < \gamma \leq \sqrt{3} - 1$ ; (2)  $\sqrt{3} - 1 < \gamma < 1$  and  $p^n < p^c \leq (\alpha(1 - \gamma)(2 + \gamma) + \gamma c) / (2 - \gamma^2)$ ; and (3)  $\sqrt{3} - 1 < \gamma < 1$  and  $(\alpha(1 - \gamma)(2 + \gamma) + \gamma c) / (2 - \gamma^2) < p^c \leq p^m$ . In the first and second case, the defector's optimal deviation price is  $(\alpha(1 - \gamma) + \gamma p^c + c) / 2$  and thus the deviation profit is  $\pi_i^r = (\alpha(1 - \gamma) + \gamma p^c - c)^2 / (4\beta(1 - \gamma^2))$ ; in the last case, its optimal deviation price is  $(p^c - \alpha(1 - \gamma)) / \gamma$  and the deviation profit is  $\pi_i^r = (-\alpha(1 - \gamma) - c\gamma + p^c)(\alpha - p^c) / (\beta\gamma^2)$ . The collusive profit is

$$\pi_i^c = (p^c - c)(\alpha - p^c) / (\beta(1 + \gamma))$$

and the Nash equilibrium profit is

$$\pi_i^n = (\alpha - c)^2 (1 - \gamma) / (\beta(1 + \gamma)(2 - \gamma^2)).$$

Substituting  $\pi_i^r$ ,  $\pi_i^c$  and  $\pi_i^n$  into the formula of critical discount factor under Nash-reversion  $\delta^c = (\pi_i^r - \pi_i^c) / (\pi_i^r - \pi_i^n)$ , we obtain the critical discount factor under Nash-reversion for each case. In the first and second case,

$$\delta^c = \frac{(2 - \gamma)^2 ((2 - \gamma)p^c - (1 - \gamma)\alpha - c)}{(2 - \gamma)\gamma^2 p^c + \gamma(1 - \gamma)(4 - \gamma)\alpha - \gamma(4 - 3\gamma)c}. \quad (11)$$

In the last case,

$$\delta^c = \frac{(2 - \gamma)^2 ((1 + \gamma - \gamma^2)p^c - (1 - \gamma^2)\alpha - \gamma c)(\alpha - p^c)}{(1 + \gamma)(2 - \gamma)^2 (p^c - \alpha(1 - \gamma) - \gamma c)(\alpha - p^c) - \gamma^2(1 - \gamma)(\alpha - c)^2}. \quad (12)$$

The critical discount factor necessary to sustain collusion under price-matching punishments is defined by  $\delta \geq \delta^c$  where

$$\delta^c = ((2 - \gamma)p^c - \alpha(1 - \gamma) - c) / (\gamma(p^c - c)). \quad (13)$$

Comparing (11) and (12) with (13), it can be shown that for our linear demand model, the set of discount factors and collusive prices sustainable under price-matching punishments

is a proper subset of those sustainable under Nash-reversion, as suggested by Proposition 5 in the text. Figure 1 in the text indicates the difference between the two regions can be large. It indicates that tacit collusion may be substantially harder to maintain under price-matching punishments, particularly if products are highly substitutable.

### 3 Appendix C: T-period punishments

In this appendix, we show our analysis continues to apply when the punishment strategy under consideration is that the punishment only lasts  $T$  periods after either firm lowers their price below the initial collusive price, after which the firms return to setting the initial collusive price  $p^c$ . The proposed strategy is defined as follows: if a firm learns that its rival set a lower price than it did in the previous period, it matches that price for  $T$  periods and then reverts to the collusive price  $p^c$ . Otherwise, the firm keeps its price unchanged. If prices in the previous period are  $p_i$  and  $p_j$ , then according to this strategy, both firms will set prices to  $\min(p_i, p_j)$  in the current period and subsequent  $T - 1$  periods, followed by a return of their prices to the collusive price  $p^c$ . (If either firm prices below the one-shot Nash equilibrium price, then as before the strategy is modified so that the firms will both set the one-shot Nash price for the  $T$  periods, followed by a return to the initial collusive price). This price-matching punishment rule is repeated in every period. This means there are two phases. The game always starts in the collusive phase with the collusive price  $p^c$ . If either firm prices below the equilibrium pricing strategy, then the punishment phase starts, which lasts for  $T$  periods, after which the firms return to the collusive phase if they follow the equilibrium pricing strategy.<sup>2</sup> As in Section 2, we have

**Proposition 1** *When the firms' products are homogenous and they face constant marginal costs (which we normalize to zero), no price above  $p^n$  is supportable by  $T$ -period price-matching punishments.*

**Proof.** If firms stick to the initial price, they each receive  $\pi_C = p^c Q(p^c) / (2(1 - \delta))$ . If either defects, setting a price of  $p^d = p^c - \varepsilon$ , it will get profits of

$$\pi_D = (p^c - \varepsilon) Q(p^c - \varepsilon) + (\delta - \delta^{T+1}) (p^c - \varepsilon) Q(p^c - \varepsilon) / (2(1 - \delta)) + \delta^{T+1} p^c Q(p^c) / (2(1 - \delta))$$

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<sup>2</sup>Prices above  $p^c$  do not need to be considered in the equilibrium analysis since they require a simultaneous deviation by both firms.

assuming the firms thereafter follow  $T$ -period price-matching punishments. Since  $\varepsilon$  can be chosen to be arbitrarily small,  $\pi^D$  can always be made higher than  $\pi^C$  for any  $\delta < 1$  unless  $p^c = 0$ . ■

We consider the possibility of limited product substitutability and make the same assumptions as in Section 2 with respect to the stage game. With these assumptions:

**Proposition 2** *The monopoly price  $p^m$  is not supportable under  $T$ -period price-matching punishments.*

**Proof.** The gain from defecting from the monopoly price is

$$\Delta = \max_{p \leq p^m} \left[ \pi_i(p, p^m) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^m) - \frac{1}{1 - \delta} \pi(p^m) \right].$$

When evaluated at  $p = p^m$ ,  $\partial \pi_i(p, p^m) / \partial p < 0$  and  $d\pi(p) / dp = 0$ , so a sufficiently small decrease in price below the monopoly price is always profitable. ■

Next we will show that Proposition 3 in Section 2 also holds for the  $T$ -period punishment case.

**Proposition 3** *There exists some maximum collusive price supportable by  $T$ -period price-matching punishments, defined as  $\bar{p}^c$  in (15). Any price  $p$  such that  $p^n < p \leq \bar{p}^c$  is supportable by  $T$ -period price-matching punishments.*

**Proof.** We first start by deriving an inequality which will be used to prove that any initial collusive price  $p^c$  in the range  $p^n < p^c \leq \bar{p}^c$  is supportable by  $T$ -period price-matching punishments. The inequality says that any single price cut deviation from the collusive phase is not profitable for any  $p^n < p^c \leq \bar{p}^c$ . Define

$$\Delta\pi(p_i, p) = \left[ \frac{\partial \pi_i(p_i, p)}{\partial p_i} + \frac{\delta - \delta^{T+1}}{1 - \delta} \frac{d\pi(p_i)}{dp_i} \right]_{p_i=p}.$$

The fact that  $p^c$  is sustainable implies  $\Delta\pi(p_i, p^c) \geq 0$ , otherwise a firm can lower its price and increase its profit while still satisfying the constraint that  $p_i \leq p^c$ . Suppose we consider some lower collusive price  $p$  such that  $p^n \leq p < p^c$ . If firm  $i$  sets the same or a lower price, its continuing profits are

$$\pi_i(p_i, p) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_i) + \frac{\delta^{T+1}}{1 - \delta} \pi(p). \quad (14)$$

This is exactly the problem for whether a firm wishes to deviate from a collusive price  $p$ , which is lower than  $p^c$ . If we can show that  $\Delta\pi(p_i, p) \geq 0$  for any common price  $p^n \leq p < p^c$ , then firms cannot increase profits with a single price cut.

Differentiating  $\Delta\pi(p_i, p)$  with respect to  $p$  yields

$$\frac{d}{dp} (\Delta\pi(p_i, p)) = \left[ \frac{\partial^2 \pi_i(p_i, p)}{\partial p_i^2} + \frac{\partial^2 \pi_i(p_i, p)}{\partial p_i \partial p} + \frac{\delta - \delta^{T+1}}{1 - \delta} \frac{d^2 \pi(p_i)}{d^2 p_i} \right]_{p_i=p}.$$

The assumptions on the stage game imply the result that  $d(\Delta\pi(p_i, p))/dp < 0$ . Given the inequality is strict, we have that  $\Delta\pi(p_i, p) > 0$  for  $p < p^c$ . Given the concavity of the profit functions, this implies profit must be strictly lower for any reduction in the price  $p_i$  from the common price  $p < p^c$ .

Given  $\Delta\pi(p_i, p^n) > 0$ ,  $\Delta\pi(p_i, p^m) < 0$ , and  $d(\Delta\pi(p_i, p))/dp < 0$ , there exists a unique price, denoted  $\bar{p}^c$ , such that

$$\Delta\pi(p_i, \bar{p}^c) = \left[ \frac{\partial \pi_i(p_i, \bar{p}^c)}{\partial p_i} + \frac{\delta - \delta^{T+1}}{1 - \delta} \frac{d\pi(p_i)}{dp_i} \right]_{p_i=\bar{p}^c} = 0. \quad (15)$$

We are going to prove that any initial collusive price  $p^c$  in the range  $p^n < p^c \leq \bar{p}^c$  is supportable by  $T$ -period price-matching punishments using the following inequality (which follows from above):

$$\max_{p_i < p^c} \left[ \pi_i(p_i, p^c) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_i) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^c) \right] < \frac{1}{1 - \delta} \pi(p^c) \text{ for all } p^n < p^c \leq \bar{p}^c. \quad (16)$$

From the one-stage deviation principle (Fudenberg and Tirole (1991), pp.108-110) it suffices to check whether there are any histories up to some stage  $t$  where one player can gain by deviating for one period from the actions prescribed by his strategy at time  $t$  and conforming to his strategy thereafter. Now consider all possible histories to stage  $t$ . It is necessary to distinguish firms' prices in different periods. To do this, we denote firm  $i$ 's ( $i = 1, 2$ ) current period price as  $p_{i,t}$ , denote its last period price as  $p_{i,t-1}$ , and likewise for other periods. Suppose we arrive in some period where the prices of the two firms in the previous period are given as  $p_{1,t-1}$  and  $p_{2,t-1}$ .

We consider several different subcases corresponding to each distinct combination of prices that can arise (ignoring simultaneous deviations).

(1)  $p^n < p_{2,t-1} \leq p_{1,t-1} \leq p^c$  (the case with  $p_{1,t-1} \leq p_{2,t-1}$  follows by a symmetric argument)

We distinguish three scenarios:

Scenario 1: If the previous period is in the collusive phase or is in the punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p_{2,t-1} < \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is the first period in a new punishment phase), firm 1 will not want to set a price higher than

$p_{2,t-1}$ , given firm 2, according to its  $T$ -period price-matching punishment strategy, will keep its price unchanged at  $p_{2,t-1}$  in the current and subsequent  $T - 1$  periods and then both firms revert to collusion price. Similarly, firm 2 will not want to set a higher price. If firm 1 sets a lower price  $p_{1,t} < p_{2,t-1}$ , it will lead to  $T$  periods of punishment and its continuing profits are

$$\pi_1(p_{1,t}, p_{2,t-1}) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_{1,t}) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^c).$$

If it sets the same price  $p_{1,t} = p_{2,t-1}$ , its continuing profits are

$$\frac{1 - \delta^T}{1 - \delta} \pi(p_{2,t-1}) + \frac{\delta^T}{1 - \delta} \pi(p^c).$$

Thus, whether firm 1 will want to match firm 2's price, or set a lower price, is determined by whether  $\max_{p_{1,t} < p_{2,t-1}} \left[ \pi_1(p_{1,t}, p_{2,t-1}) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_{1,t}) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^c) \right]$  is less than  $\frac{1 - \delta^T}{1 - \delta} \pi(p_{2,t-1}) + \frac{\delta^T}{1 - \delta} \pi(p^c)$ . Using (16) and that  $p_{2,t-1} \leq p_{1,t-1} \leq p^c$ , it can be easily shown that

$$\max_{p_{1,t} < p_{2,t-1}} \left[ \pi_1(p_{1,t}, p_{2,t-1}) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_{1,t}) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^c) \right] < \frac{1 - \delta^T}{1 - \delta} \pi(p_{2,t-1}) + \frac{\delta^T}{1 - \delta} \pi(p^c).$$

Similarly, firm 2 will not want to set a lower price  $p_{2,t} < p_{2,t-1}$ . So both firms will follow the  $T$ -period price-matching strategy given they think the other will do so.

Scenario 2: If the previous period is the  $k$ th ( $k = 1, 2, \dots, T - 1$ ) period in a punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p_{2,t-1} \geq \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is the  $k + 1$ th period in a punishment phase), firm 1 will not want to set a price higher than  $p_{2,t-1}$ , given firm 2, according to its  $T$ -period price-matching punishment strategy, will keep its price unchanged at  $p_{2,t-1}$  in the current period and subsequent  $T - k - 1$  periods then both firms revert to collusion price if  $p_{1,t} > p_{2,t-1}$ . Similarly, firm 2 will not want to set a higher price. If firm 1 sets a lower price  $p_{1,t} < p_{2,t-1}$ , it will lead to  $T$  period punishments and its continuing profits are

$$\pi_1(p_{1,t}, p_{2,t-1}) + \frac{\delta - \delta^{T+1}}{1 - \delta} \pi(p_{1,t}) + \frac{\delta^{T+1}}{1 - \delta} \pi(p^c).$$

If it sets the same price  $p_{1,t} = p_{2,t-1}$ , its continuing profits are

$$\frac{1 - \delta^{T-k}}{1 - \delta} \pi(p_{2,t-1}) + \frac{\delta^{T-k}}{1 - \delta} \pi(p^c).$$

Again, whether firm 2 will keep its price or set a lower price is the same problem as whether firm 1 will want to match firm 2's price or set a lower price. Comparing the

expression of profits in Scenario 2 with in Scenario 1, we can find that the profit is the same if firm 1 sets a lower price while the profit in Scenario 2 is higher than in Scenario 1 if firm 1 sets the same price. Given that firm 1 has no incentive to deviate from the  $T$ -period price-matching punishment strategy in Scenario 1, it has no incentive to do so either in Scenario 2.

Scenario 3: If the previous period is the  $T$ th period in a punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p_{2,t-1} \geq \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is in collusive phase), given that the other firm will set price  $p^c$ , each firm will follow the  $T$ -period price-matching punishment strategy and set price  $p^c$  since  $p^c$  is sustainable.

(2):  $p^n = p_{2,t-1} \leq p_{1,t-1} \leq p^c$  (the case with  $p_{1,t-1} \leq p_{2,t-1}$  follows by a symmetric argument)

We also distinguish three scenarios:

Scenario 1: If the previous period is in collusive phase or it is in punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p^n < \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is the first period in a new punishment phase), then it is clear that no firm has an incentive to deviate from the punishment strategy because if any firm sets a price less than  $p^n$ , prices will revert to Nash equilibrium prices and the one-shot best response to  $p^n$  is to set price  $p^n$ . In other words, a further price cut has no one period gain and can only yield a long-term loss relative to not changing prices.

Scenario 2: If the previous period is the  $k$ th ( $k = 1, 2, \dots, T-1$ ) period in a punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p^n \geq \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is the  $k+1$ th period in a punishment phase), no firm has incentive to deviate for the same reason as in Scenario 1.

Scenario 3: If the previous period is the  $T$ th period in a punishment phase but  $\min(p_{1,t-1}, p_{2,t-1}) = p^n \geq \min(p_{1,t-2}, p_{2,t-2})$  (so the current period is in collusive phase), this case is the same as Scenario 3 in (1) above.

(3) Finally, if,  $p_{1,t-1} < p^n$  and/or  $p_{2,t-1} < p^n$ , then we assumed firms revert to the one-shot Nash equilibrium prices in the current period. Given  $T$ -period price-matching punishment strategies, firms will choose the one-shot Nash equilibrium prices in the subsequent  $T-1$  periods and then revert to collusive price  $p^c$ , which clearly defines a Nash equilibrium in the subgame since there would be no gain if a firm set a price other than  $p^n$  given the rival sets  $p^n$  during the  $T$  periods of the punishment phase. ■

Proposition 2 implies the maximum collusive price  $\bar{p}^c$  must be strictly less than the monopoly price  $p^m$ . Clearly, as before, we also will have that the sustainable region of collusive prices in  $T$ -period price-matching punishment case is smaller than in the Nash reversion case. That is, collusion is harder to sustain under  $T$ -period price-matching punishments, which is obvious given the punishment is weaker than before.

Finally, using the linear demand example of Section 3 in the text, we can obtain the maximum sustainable collusive price, which is

$$\bar{p}^c = \frac{a(1-\gamma)(1-\delta^{T+1}) + c(1-\delta\gamma - \delta^{T+1}(1-\gamma))}{2(1-\delta^{T+1}(1-\gamma)) - (1+\delta)\gamma}. \quad (17)$$

Note that the collusive price in (17) is exactly the same formula as (22) in the text, except that  $\delta$  in equation (22) in the text is replaced by  $(\delta - \delta^{T+1}) / (1 - \delta^{T+1})$ , which is strictly lower than the true  $\delta$ . Collusion is easier to sustain when  $T$  increases since

$$\frac{d\bar{p}^c}{dT} = -(\ln \delta) \frac{\gamma\delta^{T+1}(1-\delta)(1-\gamma)(a-c)}{(2(1-\delta^{T+1}(1-\gamma)) - (1+\delta)\gamma)^2} > 0.$$

Now consider the impact of product differentiation. Since

$$\frac{d\bar{p}^c}{d\gamma} = -\frac{(a-c)(1-\delta)(1-\delta^{T+1})}{(2(1-\delta^{T+1}(1-\gamma)) - (1+\delta)\gamma)^2} < 0$$

we have that, as before, increased product differentiation always makes collusion easier to sustain.