The Weinstein-Yildiz Critique and Robust Predictions with Arbitrary Payoff Uncertainty*

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Abstract

Weinstein and Yildiz (2007b) show that under a richness assumption which relaxes all common-knowledge restrictions on payoffs, every rationalizable action of every (finite) type can be selected as a uniquely rationalizable action by perturbing the higher-order beliefs (the structure theorem). Consequently, types with uniquely rationalizable actions are generic in the universal type space (generic uniqueness). This *WY critique* implies that (i) a prediction for a given type is robust (for a neighborhood of the type) if and only if it consists of all rationalizable actions for that type; (ii) selecting a prediction from the rationalizable actions is either ad hoc or unnecessary. However, their richness assumption rules out prominent applications in economic models and thus undermines their critique. In this paper, we provide an algorithm that characterizes all selections from and all robust predictions of rationalizable actions without relying on any richness assumption. By invoking the characterization, we delineate the boundary of the WY critique by further characterizing the structure theorem as well as generic uniqueness from the primitives. We also use economic examples such as Cournot competition and auctions to illustrate our approach.

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1 Introduction

Economic models typically have multiple equilibria and a large set of rationalizable actions. An important research question is how to refine the large set of outcomes to make sharp predictions. Starting from Carlsson and Van Damme (1993), the sizable literature on the global games focuses on refining predictions via perturbation of higher-order beliefs. However, Weinstein and Yildiz (2007b) (hereafter, WY) prove a striking result, which casts doubt on the methodology of the global-game literature. In particular, WY show the so-called *structure theorem* that any rationalizable action of any type can be selected as a unique prediction via perturbation of higher-order beliefs. The structure theorem implies that a prediction for a given type is robust (for a neighborhood of the type) if and only if it consists of all rationalizable actions for that type. Consequently, no robust prediction refines the often permissive notion of rationalizability. Combined with the upper hemicontinuity of the rationalizability correspondence (Dekel, Fudenberg, and Morris (2006)), the structure theorem also implies *generic uniqueness*, namely that the set of types with uniquely rationalizable actions is generic (i.e., open and dense) in the universal type space. In other words, selection is unnecessary for a generic incomplete-information scenario.

The results of WY, however, rely on a "richness" assumption about the payoff uncertainty, namely that every action is strictly dominant for some payoff parameter. As WY observe, this assumption holds—in simultaneous-move games—if there is no common knowledge restriction on payoffs. Nevertheless, fixing a non-trivial dynamic game tree contradicts the richness assumption (Chen, 2012; Penta, 2012). Even a static model may impose some natural payoff structure that precludes the richness assumption. For instance, in a standard auction model, no bidder has a strictly dominant bid. In an oligopolistic competition, a relevant cost function may prevent any quantity from being strictly dominant.[1,2] The assumption therefore substantially undermines the WY critique

[1] We will scrutinize these examples in Section 3.

[2] A similar observation has been made in the global game literature. In particular, in a global game with a one-sided dominance region, we may not be able to select some action as a unique prediction (Morris and Shin, 2000; Goldstein and Pauzner, 2005; Bueno de Mesquita, 2011; Shadmehr and Bernhardt, 2012).

on the global-game literature.[3]

This paper proposes a new approach to the robustness analysis regarding perturbations on higher-order beliefs. We say that an action a_i can be *selected for a type* t_i if there exists a sequence of types $\{t_{i,m}\}$ that converges to t_i (in the product of the weak* topology) such that a_i is uniquely rationalizable for every $t_{i,m}$. Our goal is to characterize all selections of rationalizable actions for a finite type without imposing richness assumption of any kind.[4] That is, without presupposing any structure on the payoff uncertainty, we identify, for any finite type, the set of rationalizable actions that can be selected for the strategic behaviors of the type.[5] We reach our characterization in two steps:

1. We show that every finite game is intrinsically endowed with the upper ICR collection $\mathcal{R}_i^{\uparrow}$ (resp. the lower ICR collection $\mathcal{R}_i^{\downarrow}$), which is the collection of all action sets that contain (resp. are contained in) the rationalizable action set for some type;
2. Based on $\mathcal{R}_i^{\uparrow}$, we show that each finite type t_i is endowed with the local upper ICR collection $\mathcal{S}_i(t_i)$, which is the collection of all action sets that contain the rationalizable action set for some sequence of types converging to t_i.

We show that the singletons in the local upper ICR collection $\mathcal{S}_i(t_i)$ fully characterize the selections of rationalizable actions for any finite type t_i. This characterization delineates the boundary of the WY critique on the global-game literature. First, we show that the structure theorem holds if and only if every rationalizable action is uniquely rationalizable for some type. Using the upper and lower ICR collections, we can rewrite this condition as "every set in the lower ICR collection contains only actions that can be identified with singletons in the upper ICR collection." Second, we show that the generic uniqueness holds if and only if every rationalizable action set contains a uniquely rationalizable action for some type. Again, using the upper ICR collection, we can rewrite this condition

[3] There are also papers which relax the richness assumption in addressing similar but different issues. For instance, Oury and Tercieux (2012) assume a weaker richness condition and use a version of WY's argument in their study of continuous implementation. Ely and Pęski (2011) generalize generic uniqueness to the genericity of regular types without imposing the richness assumption.

[4] An extension to infinite types will be discussed in Section 5.

[5] A similar approach is also utilized in Chen, Takahashi, and Xiong (2014).

as "every set in the upper ICR collection contains some action that can be identified with a singleton in the upper ICR collection."

The local upper ICR collection also shapes the robust predictions. Specifically, we say that a prediction (a subset of rationalizable actions) for a given type is *robust for a type* t_i if it is "consistent" with ICR actions in a neighborhood of type t_i, i.e., if every nearby type of t_i has some rationalizable action that belongs to the prediction.[6,7] We show that a prediction is robust for any finite type t_i if and only if the prediction intersects every set in $\mathcal{S}_i(t_i)$.

We apply these characterizations to two economic examples and demonstrate whether to validate the WY critique. Our first example is the Cournot oligopoly game with linear inverse demand and constant marginal costs. We show that if demand is sufficiently uncertain, any quantity below the monopoly output is uniquely rationalizable for some type; otherwise, no type has a uniquely rationalizable action. In particular, if there are three firms, the Cournot oligopoly game is not dominance-solvable under complete information, but the structure theorem and generic uniqueness hold with an arbitrarily small amount of uncertainty in demand.

Our second example is a first-price auction with discrete bids and values $0, 1, \ldots, 9, 10$. In this game, any bid is rationalizable, and any bid other than 10 is uniquely rationalizable (note that bidding 10 is weakly dominated by bidding 0 for any type). Thus we can show that the structure theorem fails, but generic uniqueness holds. Moreover, we can

[6]Extending the notion of robust equilibrium in Kajii and Morris (1997) to a set-valued notion, Morris and Ui (2005) define a robust equilibrium set as a set of Nash equilibria of a complete-information game such that every nearby incomplete-information game has some Bayesian Nash equilibrium that approximates some action profile of the set. The notion of robust equilibrium set is analogous to that of robust prediction, but the former focuses on equilibrium, while the latter on rationalizability.

[7]We require consistency in a weak form, i.e., a robust prediction is required to include at least *one* rationalizable action of every nearby type, and we expect this particular action to be played only if the players can somehow coordinate their beliefs through a mediator. This is similar in spirit to the idea of partial implementation in the mechanism design literature, where at least *one* equilibrium of the mechanism is required to achieve the social choice function, and we expect this particular equilibrium to be played only via the coordination of a social planner. In the main text, however, we will discuss two other forms of robust predictions: strongly robust predictions in Definition 3 and robust* predictions in Footnote 12.

show that for the type with complete information that all bidders have values 10, all bids are rationalizable, but only bid 9 can be selected as a uniquely rationalizable action for nearby types, and $\{9\}$ is a robust prediction.

Our two-step characterization generalizes the idea in Penta (2013) that aims to propose a sufficient condition for the selections. Specifically, Penta (2013) assumes that every player has some dominant actions, which generate uniquely rationalizable actions in the universal space (corresponding to our Step 1). Based on such uniquely rationalizable actions, he proposes a condition for an action to be selected for a type (corresponding to our Step 2).[8] Instead of assuming the existence of dominant actions, our Step 1 exploits the richness of possible higher-order beliefs to identify all actions that are uniquely rationalizable for some type. Our Step 2 also highlights the necessity of considering non-singleton rationalizable action sets in characterizing all selections.

2 Preliminaries

Fix a game $G = (A_i, u_i)_{i \in I}$, where each player $i \in I$ is endowed with a set of actions A_i and a payoff function u_i that depends on the action profile $a \in A := \prod_{i \in I} A_i$ and a payoff-relevant parameter $\theta \in \Theta$. Assume that I, A, and Θ are nonempty and finite sets. While we will not impose any condition on G, we state here WY's richness condition for the ease of reference:

Definition 1 *$G = (A_i, u_i)_{i \in I}$ satisfies the richness condition if for every $i \in N$ and every $a_i \in A_i$, there exists $\theta^{a_i} \in \Theta$ such that $u_i\left(\theta^{a_i}, a_i, a_{-i}\right) > u_i\left(\theta^{a_i}, a_i', a_{-i}\right)$ for every $a_i' \in A_i \setminus \{a_i\}$ and every $a_{-i} \in A_{-i}$.*

[8] The condition claimed in Penta (2013) is incorrect. Specifically, in Subsection 4.3, we provide a counterexample (Example 1) to Penta (2013, Theorem 1) and offer a fix of the mistake (Corollary 1). However, as we show by another example (Example 2), the corrected condition is still not necessary for the selections. See also Penta (2014).

For any $\pi_i \in \Delta(\Theta \times A_{-i})$, we use $BR_i(\pi_i)$ to denote the set of best replies to π_i. That is,

$$BR_i(\pi_i) = \arg\max_{a_i \in A_i} \sum_{\theta, a_{-i}} u_i(\theta, a_i, a_{-i}) \pi_i[\theta, a_{-i}].$$

A *model* is a tuple (T, κ) where $T = \prod_{i \in I} T_i$ is a type space which associates a belief $\kappa_{t_i} \in \Delta(\Theta \times T_{-i})$ for each type $t_i \in T_i$.[9] Assume that $t_i \mapsto \kappa_{t_i}$ is a continuous mapping. Given a type $t_i \in T_i$, we can compute the first-order belief of t_i (i.e., his belief about Θ) by setting t_i^1 equal to the marginal distribution of κ_{t_i} on Θ. We can also compute the second-order belief of t_i (i.e., his belief about (θ, t_{-i}^1)) by setting

$$t_i^2[E] = \kappa_{t_i}\left[\left\{(\theta, t_{-i}) : \left(\theta, t_{-i}^1\right) \in E\right\}\right]$$

for every measurable $E \subset \Theta \times (\Delta(\Theta))^{|I|-1}$. We can compute the entire hierarchy of beliefs $(t_i^1, t_i^2, \ldots, t_i^n, \ldots)$ by proceeding in this way. A model is said to be finite if $|T| < \infty$.

We collect all such hierarchies and construct the universal type space T_i^*. This has the property that $t_i = (t_i^1, t_i^2, \ldots) \in T_i^*$ if and only if there exists some type t_i' in some model such that $t_i^n = (t_i')^n$ for every n. Endowed with the product topology, T_i^* is a compact metrizable space and admits a homeomorphism $\kappa_i^* \colon T_i^* \to \Delta(\Theta \times T_{-i}^*)$ (Mertens and Zamir, 1985). Thus, we can regard (T^*, κ^*) as a model, where $\kappa_{t_i}^* := \kappa_i^*(t_i)$ for every $t_i \in T_i^*$. Moreover, the hierarchy of beliefs of $t_i \in T_i^*$ in the model (T^*, κ^*) is given by t_i itself, and this is why we use t_i^n to denote both the n-th component of t_i and the n-th order belief of t_i in (T^*, κ^*). A type $t_i \in T_i^*$ is said to be a *finite type* if there exists a finite model (T, κ) and a type $t_i' \in T_i$ such that t_i' has the hierarchy of beliefs t_i. With a further abuse of notations, we say that a sequence of types $\{t_{i,m}\}_{m=0}^{\infty}$ on T_i^* converges to a type t_i in some (not necessarily the universal) model, denoted as $t_{i,m} \to t_i$, if for every n, $t_{i,m}^n \to t_i^n$ in the weak* topology as $m \to \infty$.

Let (T, κ) be a model. We define the solution concept of interim correlated rationalizability (ICR) (Dekel, Fudenberg, and Morris, 2006, 2007) as follows. For $i \in I$ and type

[9] Throughout the paper, for any metrizable space Y, we use $\Delta(Y)$ to denote the space of probability measures on the Borel σ-algebra of Y. We endow $\Delta(Y)$ with the weak* topology. Moreover, we endow a product space with the product topology, a subspace with the relative topology, and a finite set with the discrete topology. Let $|E|$ denote the cardinality of a set E.

$t_i \in T_i$, set $ICR_i^0(t_i) = A_i$; define sets $ICR_i^n(t_i)$ for $n > 0$ iteratively by letting $a_i \in ICR_i^n(t_i)$ if and only if there is some *conjecture* $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ such that

(i) $\operatorname{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$;

(ii) $\nu_i \left[\left\{ (\theta, t_{-i}, a_{-i}) : a_{-i} \in ICR_{-i}^{n-1}(t_{-i}) \right\} \right] = 1$;

(iii) $a_i \in BR_i \left(\operatorname{marg}_{\Theta \times A_{-i}} \nu_i \right)$.

Then, define

$$ICR_i(t_i) = \bigcap_{n=0}^{\infty} ICR_i^n(t_i).$$

We write $ICR_{-i}^{n-1}(t_{-i}) = \prod_{j \neq i} ICR_j^{n-1}(t_j)$ and $ICR_{-i}(t_{-i}) = \prod_{j \neq i} ICR_j(t_j)$. Call conjecture $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ *valid for* t_i if $\operatorname{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$ and $\nu_i [a_{-i} \in ICR_{-i}(t_{-i})] = 1$. Dekel, Fudenberg, and Morris (2007, Proposition 4) show that

$$ICR_i(t_i) = \bigcup_{\nu_i \text{ is a valid conjecture for } t_i} BR_i \left(\operatorname{marg}_{\Theta \times A_{-i}} \nu_i \right); \tag{1}$$

moreover, $ICR_i(\cdot)$ only depends on the belief hierarchy of a type. We will hereafter identify a type with its belief hierarchy. We reproduce Chen (2012, Lemma 3) here for the sake of later use. The lemma says that for any type t_i, we can find a sequence of finite types that approximate t_i in beliefs and have the same rationalizable behaviors as t_i.

Lemma 1 *For any type $t_i \in T_i^*$, there is a sequence of finite types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $ICR_i(t_{i,m}) = ICR_i(t_i)$ for every m.*

Following WY, we now say that an action can be selected for t_i if there is a sequence of types $\{t_{i,m}\}$ converging to t_i along which a_i is uniquely rationalizable. Namely, a modeler who knows the belief of a type t_i of interest only approximately cannot preclude the possibility that a_i is a uniquely rationalizable action for some "true type" $t_{i,m}$.

Definition 2 *Given a model (T, κ), an action $a_i \in A_i$ can be selected for $t_i \in T_i$ if there is a sequence of types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $ICR_i(t_{i,m}) = \{a_i\}$ for every m.*[10]

[10] Chen, Takahashi, and Xiong (2014) use the term "WY-selection" to differentiate it from another selection notion "robust selection".

A *prediction* for a type t_i is a nonempty subset P_i of $ICR_i(t_i)$. We may think of P_i as a statement that holds if and only if the action being played lies in P_i.

We then introduce two definitions of robust prediction as follows.[11]

Definition 3 *Given a model (T, κ), a prediction $P_i \subset ICR_i(t_i)$ is robust (resp. strongly robust) for type $t_i \in T_i$ if for every sequence of types $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$, we have $P_i \cap ICR_i(t_{i,m}) \neq \varnothing$ (resp. $P_i \subset ICR_i(t_{i,m})$) for sufficiently large m.*[12]

That is, if a modeler knows the belief of a type t_i up to sufficiently high orders, then some (resp. every) element in P_i predicts a rationalizable action for "true type" $t_{i,m}$ correctly. Clearly, $ICR_i(t_i)$ is a robust prediction for t_i. Under the richness condition, WY show that every rationalizable action can be selected, and hence $ICR_i(t_i)$ is the only robust prediction for t_i, and no prediction is strongly robust if $|ICR_i(t_i)| \geq 2$.

Note that the notion of strong robustness can be derived from that of robustness. Namely, a prediction is strongly robust if and only if it consists of singleton robust predictions.

3 Examples

In this section, we present two economic examples of incomplete-information games where no player has a dominant action at any state, and thus WY's analysis cannot be applied. Nevertheless, we can "endogenize" the richness condition by identifying a large set of actions that are uniquely rationalizable for some type.

[11] See Subsection 3.2 for illustration of the notions of prediction and robust prediction.

[12] One can also define a *robust** prediction for t_i as a prediction that contains all rationalizable actions for all types close to t_i. By the upper hemicontinuity of ICR, we can show that even without the richness condition, $ICR_i(t_i)$ is the only robust* prediction for t_i.

3.1 Cournot Oligopoly with Uncertainty in Demand

Consider the Cournot oligopoly game, where the inverse demand is linear in the form of $P(Q, \theta) = \theta - Q$ with $Q = \sum_i q_i$ and parameter $\theta > 0$, and marginal costs are constant and normalized to be 0.[13] Assume that firm i can produce any nonnegative output q_i. Thus firm i's profit is given by $u_i(q_1, \ldots, q_{|I|}, \theta) = \left(\theta - \sum_j q_j\right) q_i$.

Under complete information about θ, it is well known that the Cournot oligopoly game is dominance-solvable (i.e., has a uniquely rationalizable action) if and only if $|I| = 2$ (Bernheim, 1984). Moreover, if $|I| = 2$, then the dominance solvability result extends to the case with incomplete information (Weinstein and Yildiz, 2007a, Proposition 1).[14] We will thus analyze the case where $|I| \geq 3$ and firms have incomplete information about θ. For simplicity, we assume that θ takes two possible values, θ_H and θ_L with $\theta_H > \theta_L > 0$.[15]

We denote by $\mathbb{E}_{t_i^1}(\theta)$ the expected value of θ with respect to the first-order belief t_i^1 of type t_i.

Proposition 1 *Consider the Cournot oligopoly game with $|I| \geq 3$ firms and uncertainty in demand.*

(a) Suppose that $\theta_H/\theta_L > (|I| - 1)/2$. Then action q is uniquely rationalizable for some type in T_i^* if and only if $q \in [0, \theta_H/2]$.

(b) Suppose that $\theta_H/\theta_L \leq (|I| - 1)/2$. Then we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$ for any $t_i \in T_i^*$; in particular, no type has a uniquely rationalizable action.

To see how the condition on θ_H/θ_L is used in the proof of part (a), consider a type $\tau_{i,1,H}$ who is certain that "$\theta = \theta_H$ and each opponent $j \neq i$ is certain about $\theta = \theta_L$." Since

[13] We allow for negative prices which only mean that the demand function is linear in prices even below marginal costs.

[14] Weinstein and Yildiz (2011) study the sensitivity of *equilibrium* behavior to higher-order beliefs in Cournot games. In contrast, here we focus on selecting rationalizable actions as uniquely rationalizable actions (see also Subsection 4.5.1). The notion of convergence of higher-order beliefs that Weinstein and Yildiz (2011) consider is also slightly stronger than the product convergence that WY and we consider here.

[15] A similar exercise can be done with uncertainty in cost functions.

$\tau_{i,1,H}$ believes that each $j \neq i$ plays an action of at most $\theta_L/2$, the action that $\tau_{i,1,H}$ can rationalize is at least

$$\frac{1}{2}\left(\theta_H - (|I|-1)\frac{\theta_L}{2}\right),$$

which is strictly positive since $\theta_H/\theta_L > (|I|-1)/2$. Similarly, we consider the type $\tau_{i,2,L}$ who is certain that "$\theta = \theta_L$ and each opponent $j \neq i$ is of type $\tau_{j,1,H}$." Then the action that $\tau_{i,2,L}$ can rationalize is at most

$$\frac{1}{2}\left(\theta_L - (|I|-1)\frac{1}{2}\left(\theta_H - (|I|-1)\frac{\theta_L}{2}\right)\right),$$

which is strictly below $\theta_L/2$. Continuing these processes alternatingly sufficiently many times, we can construct a type for which action 0 is uniquely rationalizable. Then the final step of the proof is to extend this result to any action in $[0, \theta_H/2]$. See Appendix A.1 for a more formal proof.

Proposition 1 exhibits a sharp discontinuity: (a) if θ_H/θ_L is large, then any action that is rationalizable for some type is uniquely rationalizable for some other type; (b) if θ_H/θ_L is small, then no type has a uniquely rationalizable action. In particular, if $|I| = 3$, with an arbitrarily small amount of uncertainty in demand, we have $\theta_H/\theta_L > 1 = (|I|-1)/2$, and Proposition 1(a) applies. This is in contrast with the case under complete information, where the Cournot oligopoly game is not dominant-solvable.

Note that Proposition 1 continues to hold for finely discretized action spaces. For example, suppose that firms can produce outputs only in $d\mathbb{N}$, the set of nonnegative integer multiples of $d > 0$. Assume $\theta_L/2 \in d\mathbb{N}$ for simplicity. Then, (a) if $\theta_H/\theta_L > (|I|-1)/2 + d/\theta_L$, then any action in $[0, (\theta_H + d)/2] \cap d\mathbb{N}$ is uniquely rationalizable for some type in T_i^*; (b) if $\theta_H/\theta_L \leq (|I|-1)/2 + d/\theta_L$, then we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta + d)/2\right] \cap d\mathbb{N}$.

3.2 First-Price Auction with Discrete Bids

Consider a sealed-bid first-price auction with $|I| \geq 3$, where bidders submit their bids $b_1, \ldots, b_{|I|} \in \{0, 1, \ldots, 9, 10\}$ simultaneously. Tie breaking is based on a fair coin toss. Each bidder's value for the object is in $\{0, 1, \ldots, 9, 10\}$, i.e., $\Theta = \{0, 1, \ldots, 9, 10\}^I$. Observe that in this example, no bid is strictly dominant for any value and thus WY's richness condition does not hold.

We show that bidding b is uniquely rationalizable for some type in T_i^* if and only if $b \neq 10$. To see the "if" direction, let $\tau_{i,0}$ be the type with complete information that all bidders have values 0. It is easy to see that $ICR_i(\tau_{i,0}) = \{0\}$. We construct types $\tau_{i,b} \in T_i^*$ with $ICR_i(\tau_{i,b}) = \{b\}$ inductively. For any $b \in \{1, \ldots, 9\}$, let $\tau_{i,b}$ be the type of bidder i who is certain that his own value is $b+1$ and $t_j = \tau_{j,b-1}$ for $j \neq i$. Then, by the induction hypothesis, type $\tau_{i,b}$ believes that the opponents bid $b-1$. Since $|I| \geq 3$, bidding b is the unique best response. By (1), we have $ICR_i(\tau_{i,b}) = \{b\}$. The "only if" direction is immediate since any type who believes that his own value is 10 is indifferent between bidding 0 and 10, and any other type strictly prefers bidding 0.

We also illustrate the notion of robust prediction in this example. Let $\tau_{i,10}$ be the type with complete information that all bidders have values 10. Then, we have $ICR_i(\tau_{i,10}) = \{0, 1, \ldots, 9, 10\}$ since every bid is a best reply to the belief that the opponents bid 10. On the other hand, since bidding 10 is weakly dominated by bidding 0 for $\tau_{i,10}$, $\{0, 1, \ldots, 9\}$ is a robust prediction for $\tau_{i,10}$.

4 Main Results

4.1 The Upper and Lower ICR Collections

We denote by $\mathcal{A}_i$ the collection of all nonempty subsets of A_i. For each $(\mathcal{B}_j)_{j \neq i}$ with $\mathcal{B}_j \subset \mathcal{A}_j$, we denote by $\mathcal{B}_{-i}$ the collection of all product sets $B_{-i} = \prod_{j \neq i} B_j$ with $B_j \in \mathcal{B}_j$. Say that $\pi_i \in \Delta(\Theta \times A_{-i})$ is *consistent with* $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$ if there exists a function $\varphi_i \colon \Theta \times \mathcal{A}_{-i} \to \Delta(A_{-i})$ such that

$$\varphi_i(\theta, R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in R_{-i}; \tag{2}$$

$$\pi_i[\theta, a_{-i}] = \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i[\theta, R_{-i}] \varphi_i(\theta, R_{-i})[a_{-i}]. \tag{3}$$

For a given $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$, we denote by $\Pi_i^{\mu_i}$ the set of π_i's that are consistent with μ_i. Namely, $\Pi_i^{\mu_i}$ is the set of player i's beliefs over $\Theta \times A_{-i}$ if state θ and a nonempty product set of actions R_{-i} realize according to μ_i, and the opponents choose actions (possibly stochastically) from R_{-i}.[16]

We define the *upper ICR collection* $\mathcal{R}_i^{\uparrow}$ and the *lower ICR collection* $\mathcal{R}_i^{\downarrow}$ as follows:

$$\mathcal{R}_i^{\uparrow} := \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \supset ICR_i(t_i)\},$$
$$\mathcal{R}_i^{\downarrow} := \{R_i \in \mathcal{A}_i : \exists t_i \in T_i^* \text{ s.t. } R_i \subset ICR_i(t_i)\}.$$

Note that identifying $\mathcal{R}_i^{\uparrow}$ is equivalent to identifying all minimal ICR sets; identifying $\mathcal{R}_i^{\downarrow}$ is equivalent to identifying all maximal ICR sets.

Both the upper and lower ICR collections will play important roles in our characterization results. For example, we will show that the structure theorem holds if and only if every rationalizable action is uniquely rationalizable for some type. Using $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$, we can rewrite the latter condition as "for any $i \in I$, $a_i \in A_i$, and $R_i \in \mathcal{R}_i^{\downarrow}$, if $a_i \in R_i$, then $\{a_i\} \in \mathcal{R}_i^{\uparrow}$" (see Corollary 2 in Subsection 4.3).

We now provide algorithms to compute $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ from the primitives.[17] For the algorithm to compute $\mathcal{R}_i^{\uparrow}$, let $\mathcal{R}_i^{\uparrow,0} := \{A_i\}$ for each $i \in I$. For each $i \in I$ and $n \geq 1$, we define $\mathcal{R}_i^{\uparrow,n}$ inductively as follows:

$$\mathcal{R}_i^{\uparrow,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow,n-1}\right) \text{ s.t. } R_i \supset \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

[16] Each μ_i imposes finitely many linear inequality constraints on π_i since $\pi_i \in \Pi_i^{\mu_i}$ if and only if $\sum_{a_{-i} \in B_{-i}} \pi_i[\theta, a_{-i}] \geq \sum_{R_{-i} \subset B_{-i}} \mu_i[\theta, R_{-i}]$ for any $\theta \in \Theta$ and $B_{-i} \subseteq A_{-i}$ (Strassen, 1964).

[17] We are not aware of any finite-step finite-dimensional algorithm to compute all (not necessarily minimal or maximal) ICR sets. For example, if we let $\mathcal{R}_i^{=,0} := \{A_i\}$ and

$$\mathcal{R}_i^{=,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{=,n-1}\right) \text{ s.t. } R_i = \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\},$$

we do not know if the algorithm terminates in finite steps or not; the sequence $\mathcal{R}_i^{=,n}$ may enter a repeating cycle. Fortunately, in order to characterize all selections, (strongly) robust predictions, the structure theorem, and generic uniqueness, it is enough to use $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$.

First, note that each step is a finite-dimensional and linear problem. Second, it is without loss of generality that μ_i puts positive probabilities only on minimal sets in $\mathcal{R}_{-i}^{\uparrow,n-1}$.[18] Third, observe that $\mathcal{R}_i^{\uparrow,n}$ is increasing in the set-inclusion order, i.e., $\mathcal{R}_i^{\uparrow,0} \subset \mathcal{R}_i^{\uparrow,1} \subset \mathcal{R}_i^{\uparrow,2} \subset \cdots$. Moreover, $\mathcal{R}_i^{\uparrow,n'} = \mathcal{R}_i^{\uparrow,n}$ for all $i \in I$ and $n' \geq n$ whenever $\mathcal{R}_i^{n} = \mathcal{R}_i^{n-1}$ for all $i \in I$. Therefore, the computation takes at most $\sum_i 2^{|A_i|} - 2\,|I|$ steps.

For the algorithm to compute $\mathcal{R}_i^{\downarrow}$, let $\mathcal{R}_i^{\downarrow,0} := \mathcal{A}_i$ for each $i \in I$. For each $i \in I$ and $n \geq 1$, we define $\mathcal{R}_i^{\downarrow,n}$ inductively as follows:

$$\mathcal{R}_i^{\downarrow,n} := \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\downarrow,n-1}\right) \text{ s.t. } R_i \subset \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

Symmetrically to $\mathcal{R}_i^{\uparrow,n}$, $\mathcal{R}_i^{\downarrow,n}$ is decreasing and reaches its limit in at most $\sum_i 2^{|A_i|} - 2\,|I|$ steps.

The next proposition shows that from the primitives (i.e., the fixed game $G = (A_i, u_i)_{i \in I}$), we can obtain $\mathcal{R}_i^{\uparrow}$ (resp. $\mathcal{R}_i^{\downarrow}$) by computing $\mathcal{R}_i^{\uparrow,n}$ (resp. $\mathcal{R}_i^{\downarrow,n}$) in finitely many steps (see Appendix A.2 for the proof). Thus, we will subsequently take $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ as given.

Proposition 2 *For any $n \geq \sum_i 2^{|A_i|} - 2\,|I|$, we have (a) $\mathcal{R}_i^{\uparrow,n} = \mathcal{R}_i^{\uparrow}$; (b) $\mathcal{R}_i^{\downarrow,n} = \mathcal{R}_i^{\downarrow}$.*

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[18] Therefore, instead of $\mathcal{R}_i^{\uparrow,n}$, we could have defined $\mathcal{R}_i^{\min,n}$ by letting $\mathcal{R}_i^{\min,0} := \{A_i\}$ and

$$\mathcal{R}_i^{\min,n} := \text{all minimal sets of } \left\{ R_i \in \mathcal{A}_i : \exists \mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\min,n-1}\right) \text{ s.t. } R_i = \bigcup_{\pi_i \in \Pi_i^{\mu_i}} BR_i(\pi_i) \right\}.$$

$\mathcal{R}_i^{\uparrow,n}$ is increasing in the set-inclusion order, whereas $\mathcal{R}_i^{\min,n}$ becomes "finer" in a more convoluted sense. Similar comments apply to $\mathcal{R}_i^{\downarrow,n}$ in this subsection and $\mathcal{S}_i^{n}(t_i)$ in Subsection 4.2.

4.2 The Local Upper ICR Collection and Characterizations of Selections and Robust Predictions

Fix a finite model (T, κ). Say that $\pi_i \in \Delta(\Theta \times A_{-i})$ is *consistent with* $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$ if there exists a function $\varphi_i \colon \Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i} \to \Delta(A_{-i})$ such that

$$\varphi_i(\theta, t_{-i}, R_{-i})[a_{-i}] > 0 \Rightarrow a_{-i} \in R_{-i}; \tag{4}$$

$$\pi_i[\theta, a_{-i}] = \sum_{t_{-i}, R_{-i}} \mu_i\left[\theta, t_{-i}, R_{-i}\right] \varphi_{-i}(\theta, t_{-i}, R_{-i})[a_{-i}]. \tag{5}$$

For a given $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$, with a slight abuse of notations, we also denote by $\Pi_i^{\mu_i}$ the set of all conjectures that are consistent with μ_i.[19]

In order to characterize all selections, for each type $t_i \in T_i^*$, we define the *local upper ICR collection* $\mathcal{S}_i^*\left(t_i\right)$ *for* t_i as follows:

$$\mathcal{S}_i^*\left(t_i\right) := \left\{R_i \in \mathcal{A}_i : \exists \left\{t_{i,m}\right\}_{m=0}^{\infty} \subset T_i^* \text{ s.t. } t_{i,m} \to t_i \text{ and } R_i \supset ICR_i\left(t_{i,m}\right), \forall m\right\}.$$

Then, characterizing actions that can be selected for t_i amounts to determining the singletons in $\mathcal{S}_i^*\left(t_i\right)$. We now define an algorithm that can be used to "solve" $\mathcal{S}_i^*\left(t_i\right)$ in finitely many steps.

For each $i \in I$ and $t_i \in T_i$, let $\mathcal{S}_i^0\left(t_i\right) := \mathcal{R}_i^{\uparrow}$, and for each $n \geq 1$, define

$$\mathcal{S}_i^n\left(t_i\right) := \left\{ R_i \in \mathcal{A}_i : \begin{array}{l} \forall \varepsilon \in (0,1], \exists \left(\mu_i, \mu_i'\right) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right) \times \Delta\left(\Theta \times \mathcal{R}^{\uparrow}_{-i}\right) \text{ s.t.} \\ \text{(i) } \operatorname{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}; \\ \text{(ii) } \mu_i\left[\left\{\left(\theta, t_{-i}, R_{-i}\right) : R_{-i} \in \mathcal{S}_{-i}^{n-1}\left(t_{-i}\right)\right\}\right] = 1; \\ \text{(iii) } R_i \supset \bigcup_{\left(\pi_i, \pi_i'\right) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}} BR_i\left(\left(1-\varepsilon\right)\pi_i + \varepsilon \pi_i'\right) \end{array} \right\}. \tag{6}$$

Note that each step is a semialgebraic problem, i.e., a problem based on finitely many variables and polynomial equations and inequalities. Also, it is without loss of generality that μ_i and μ_i' put positive probabilities only on minimal sets in $S_{-i}^{n-1}(t_{-i})$ and in $\mathcal{R}^{\uparrow}_{-i}$ (i.e., minimal ICR sets), respectively. Moreover, $\mathcal{S}_i^n\left(t_i\right)$ is decreasing, and reaches its limit, denoted by $\mathcal{S}_i\left(t_i\right)$, in at most $\sum_i \left(|\mathcal{R}_i^{\uparrow}| - 1\right) |T_i|$ steps. Put differently, $\mathcal{S}_i\left(t_i\right)$ is the

[19] In particular, we will use $\Pi_i^{\mu_i}$ for both $\mu_i \in \Delta\left(\Theta \times \mathcal{A}_{-i}\right)$ and $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$.

largest (i.e., finest) profile of sub-collections of $\mathcal{R}_i^{\uparrow}$ that satisfies the following fixed-point property:

$$\mathcal{S}_i\left(t_i\right)=\left\{R_i\in\mathcal{A}_i:\begin{array}{l}\forall\varepsilon\in(0,1],\exists\left(\mu_i,\mu_i'\right)\in\Delta\left(\Theta\times T_{-i}\times\mathcal{R}_{-i}^{\uparrow}\right)\times\Delta\left(\Theta\times\mathcal{R}_{-i}^{\uparrow}\right)\text{ s.t.}\\ \text{(i) }\operatorname{marg}_{\Theta\times T_{-i}}\mu_i=\kappa_{t_i};\\ \text{(ii) }\mu_i\left[\left\{\left(\theta,t_{-i},R_{-i}\right):R_{-i}\in\mathcal{S}_{-i}\left(t_{-i}\right)\right\}\right]=1;\\ \text{(iii) }R_i\supset\bigcup_{\left(\pi_i,\pi_i'\right)\in\Pi_i^{\mu_i}\times\Pi_i^{\mu_i'}}BR_i\left(\left(1-\varepsilon\right)\pi_i+\varepsilon\pi_i'\right)\end{array}\right\}. \tag{7}$$

Formally, we obtain the following result

Proposition 3 *$\mathcal{S}_i\left(t_i\right)=\mathcal{S}_i^*\left(t_i\right)$ for finite type t_i.*

See Appendix A.3 for the proof. We prove one direction $\mathcal{S}_i\left(t_i\right)\subset\mathcal{S}_i^*\left(t_i\right)$ by exploiting the fixed point property of $\mathcal{S}_i(t_i)$ in (7) and for each $R_i\in\mathcal{S}_i\left(t_i\right)$, constructing types $\{t_{i,m}\}$ with $t_{i,m}\to t_i$ and $R_i\supset ICR_i(t_{i,m})$. The other direction $\mathcal{S}_i^*\left(t_i\right)\subset\mathcal{S}_i\left(t_i\right)$ follows from establishing that $\mathcal{S}_i^*\left(t_i\right)$ also satisfies the same fixed point property.[20] Intuitively speaking, iteration in $\mathcal{S}_i^n(t_i)$ is to match $t_{i,m}$ with the limit type t_i up to the n-th order, and ε in (6) and (7) corresponds to perturbations in beliefs at each order.

Recall that an action can be selected for a type t_i if there is a sequence of types $\{t_{i,m}\}_{m=0}^{\infty}\subset T_i^*$ such that $t_{i,m}\to t_i$ and $ICR_i\left(t_{i,m}\right)=\{a_i\}$ for every m. Also recall that a prediction $P_i\subset ICR_i\left(t_i\right)$ is robust (resp. strongly robust) for type $t_i\in T_i$ if for every sequence of types $\{t_{i,m}\}_{m=0}^{\infty}\subset T_i^*$ such that $t_{i,m}\to t_i$, we have $P_i\cap ICR_i\left(t_{i,m}\right)\neq\varnothing$ (resp. $P_i\subset ICR_i\left(t_{i,m}\right)$) for sufficiently large m. The following theorems, which follow immediately from Proposition 3, show that $\mathcal{S}_i\left(t_i\right)$ contains enough information to characterize all selections and robust predictions for t_i.

Theorem 1 *Action a_i can be selected for finite type t_i if and only if $\{a_i\}\in\mathcal{S}_i\left(t_i\right)$.*

Theorem 2 *Prediction P_i is robust (resp. strongly robust) for finite type t_i if and only if $P_i\cap R_i\neq\varnothing$ (resp. $P_i\subset R_i$) for any $R_i\in\mathcal{S}_i\left(t_i\right)$.*

[20] We will employ the fixed-point property to define perturbed curb collections when we consider infinite types in Section 5.

4.3 A Reduction to Singletons

The local upper ICR collection for t_i captures all action sets that contain the ICR set for some sequence of types converging to t_i. Also, the algorithm of computing the local upper ICR collection stops in finitely many steps, and each step is a finite-dimensional problem. However, unlike conventional algorithms in game theory (such as the algorithm of computing all rationalizable actions in a complete-information game), our algorithm involves probabilities over collections of action sets, which may appear complicated at first glance. In this subsection, we simplify our algorithm by reducing the $\mathcal{S}_i^n(t_i)$ sequence to collections of singletons. We also compare the simplified algorithm with the sufficient condition for the selections claimed in Penta (2013).

Let R_i^{u} (where superscript u stands for uniqueness) be the set of all actions that are uniquely rationalizable for some type:

$$R_i^{\mathrm{u}} := \left\{a_i \in A_i \mid \{a_i\} \in \mathcal{R}_i^{\uparrow}\right\}.$$

Given a finite model (T, κ), for each $i \in I$ and $t_i \in T_i$, let $S_i^{\mathrm{u},0}(t_i) := ICR_i(t_i) \cap R_i^{\mathrm{u}}$, and for each $n \geq 1$, define

$$S_i^{\mathrm{u},n}(t_i) := \left\{ a_i \in R_i^{\mathrm{u}} : \begin{array}{l} \exists \mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times T_{-i} \times R_{-i}^{\mathrm{u}}\right) \text{ s.t.} \\ \text{(i) } \mathrm{marg}_{\Theta \times T_{-i}} \mu_i^{\mathrm{u}} = \kappa_{t_i}; \\ \text{(ii) } \mu_i^{\mathrm{u}}\left[\left\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in S_{-i}^{\mathrm{u},n-1}(t_{-i})\right\}\right] = 1; \\ \text{(iii) } a_i \in BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \mu_i^{\mathrm{u}}\right) \end{array} \right\}. \tag{8}$$

We have $R_i^{\mathrm{u}} = S_i^{\mathrm{u},0}(t_i) \supset S_i^{\mathrm{u},1}(t_i) \supset \cdots$, which reaches its limit $S_i^{\mathrm{u}}(t_i)$ in finitely many steps.[21]

Corollary 1 *Action a_i can be selected for finite type t_i if $a_i \in S_i^{\mathrm{u}}(t_i)$.*

Proof By Theorem 1, it suffices to show that $a_i \in S_i^{\mathrm{u}}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i(t_i)$. We prove by induction that $a_i \in S_i^{\mathrm{u},n}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i^n(t_i)$. The case for $n = 0$ holds by

[21] Note that $S_i^{\mathrm{u}}(t_i)$ can be empty. More precisely, $S_i^{\mathrm{u}}(t_i) = \varnothing$ if and only if $ICR_j(t_j) \cap R_j^{\mathrm{u}} = \varnothing$ for some t_j in the smallest belief-closed type space containing t_i. In this case, Corollary 1 is vacuously true, and we should instead apply Theorem 1.

definition. Now suppose that $a_i \in S_i^{\mathrm{u},n-1}(t_i)$ implies $\{a_i\} \in \mathcal{S}_i^{n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Let $a_i \in S_i^{\mathrm{u},n}(t_i)$ and we show that $\{a_i\} \in \mathcal{S}_i^{n}(t_i)$. Since $a_i \in S_i^{\mathrm{u},n}(t_i)$, there exists $\mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times T_{-i} \times R_{-i}^{\mathrm{u}}\right)$ that satisfies (i)-(iii) in (8). Moreover, since $a_i \in R_i^{\mathrm{u}}$, there exists $t_i' \in T_i^*$ such that $\{a_i\} = ICR_i(t_i')$. By (1), we have $\{a_i\} = BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i'\right)$ for any valid conjecture $\nu_i' \in \Delta(\Theta \times T_{-i}^* \times A_{-i})$ for t_i'. Define $(\mu_i, \mu_i') \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that

$$\mu_i[\theta, t_{-i}, \{a_{-i}\}] = \mu_i^{\mathrm{u}}[\theta, t_{-i}, a_{-i}];$$
$$\mu_i'[\theta, R_{-i}] = \kappa_{t_i'}^*[\{(\theta, s_{-i}) : ICR_{-i}(s_{-i}) = R_{-i}\}]$$

for each $(\theta, t_{-i}, a_{-i}, R_{-i}) \in \Theta \times T_{-i} \times A_{-i} \times \mathcal{R}_{-i}^{\uparrow}$. Then, μ_i satisfies (i) and (ii) in (6) because μ_i^{u} satisfies (i) and (ii) in (8) and we assume the induction hypothesis. It follows from (iii) in (8) and $\{a_i\} = BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i'\right)$ for any valid conjecture ν_i' for t_i' that $\{a_i\} = BR_i\left((1-\varepsilon)\pi_i + \varepsilon \pi_i'\right)$ for every $(\pi_i, \pi_i') \in \Pi_i^{\mu_i} \times \Pi_i^{\mu_i'}$. Thus, $\{a_i\} \in \mathcal{S}_i^n(t_i)$. ■

Observe that under the richness condition, $R_i^{\mathrm{u}} = A_i$, and therefore $S_i^{\mathrm{u},n}(t_i) = ICR_i^n(t_i)$ for every n and $S_i^{\mathrm{u}}(t_i) = ICR_i(t_i)$. Thus, Corollary 1 immediately reproduces WY's result that every ICR action can be selected for every finite type, under the richness condition.

To compare Corollary 1 with Penta (2013, Theorem 1), we recap his analysis as follows. First, let $\mathcal{A}_i^0$ be the set of actions of player i that is strictly dominant in some θ. Then define

$$\mathcal{A}_i^n := \left\{a_i \in A_i : \exists \mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times \mathcal{A}_{-i}^{n-1}\right) \text{ s.t. } \{a_i\} = BR_i(\mu_i^{\mathrm{u}})\right\},$$

and $\mathcal{A}_i^\infty = \bigcup_{n \geq 0} \mathcal{A}_i^n$. Finally, Penta's Theorem 1 states that every action in the following set can be selected for t_i:

$$ICR_i(t_i; \mathcal{A}^\infty) := \left\{ a_i \in ICR_i(t_i) \cap \mathcal{A}_i^\infty : \begin{array}{l} \exists \mu_i^{\mathrm{u}} \in \Delta\left(\Theta \times T_{-i} \times \mathcal{A}_{-i}^\infty\right) \text{ s.t.} \\ \text{(i) } \mathrm{marg}_{\Theta \times T_{-i}} \mu_i^{\mathrm{u}} = \kappa_{t_i}; \\ \text{(ii) } a_i \in BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \mu_i^{\mathrm{u}}\right) \end{array} \right\}.$$

Thus, there are two essential differences between our Corollary 1 and Penta's Theorem 1. First, R_i^{u} consists of all actions that are uniquely rationalizable for some type, whereas the definition of $\mathcal{A}_i^\infty$ starts from dominant actions $\mathcal{A}_i^0$. Obviously, R_i^{u} is larger than $\mathcal{A}_i^\infty$ and R_i^{u} can be nonempty even if players have no dominant actions in any state. Second, we define $S_i^{\mathrm{u},n}(t_i)$ recursively from $ICR_i(t_i) \cap R_i^{\mathrm{u}}$, whereas Penta takes $ICR_i(t_i; \mathcal{A}^\infty)$ without the recursion.[22] We now present two examples. The first example shows that we may not be able to select an action in $ICR_i(t_i; \mathcal{A}^\infty)$ for t_i. The second example shows that it is possible to have $\{a_i\} \in \mathcal{S}_i(t_i)$ but $a_i \notin S_i^{\mathrm{u}}(t_i)$. That is, the condition in Corollary 1 is sufficient, but not necessary for the selections.

Example 1 *Consider a game with $I = \{1,2\}$, $A_1 = \{U, D\}$, $A_2 = \{L, L', R\}$, $\Theta = \{\theta_0, \theta_1\}$, and the payoffs u_1 and u_2 are given by*

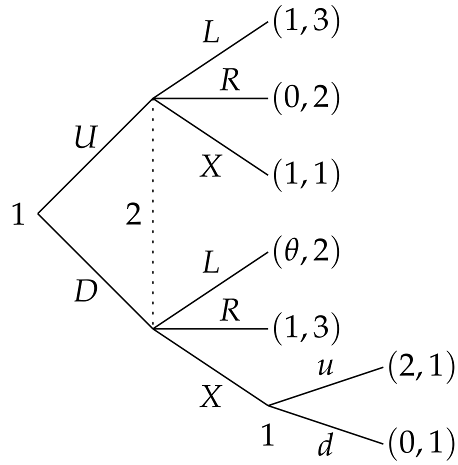
$$\theta_0 : \begin{array}{c|c|c|c|} & L & L' & R \\ \hline U & 1,1 & 0,1 & 0,0 \\ \hline D & 0,1 & 1,1 & 1,0 \\ \hline \end{array} \quad \textit{and} \quad \theta_1 : \begin{array}{c|c|c|c|} & L & L' & R \\ \hline U & 0,0 & 0,0 & 0,1 \\ \hline D & 0,0 & 0,0 & 0,1 \\ \hline \end{array}.$$

Let $\tau_{i,0}$ be the type of player i with complete information about $\theta = \theta_0$. Then it is easy to see that $ICR_1(\tau_{1,0}) = \{U, D\}$. Moreover, we have $R_1^{\mathrm{u}} = \mathcal{A}_1^\infty = \{D\}$, $R_2^{\mathrm{u}} = \mathcal{A}_2^\infty = \{R\}$, and $ICR_1(\tau_{1,0}; \mathcal{A}^\infty) = \{D\}$. Thus, Penta (2013, Theorem 1) claims that D can be selected for $\tau_{1,0}$. However, we have $\mathcal{R}_1^{\uparrow} = \{\{D\}, \{U, D\}\}$ and $\mathcal{R}_2^{\uparrow} = \{\{L, L'\}, \{R\}, \{L, L', R\}\}$. Following the algorithm in Subsection 4.2, we have $\mathcal{S}_1(\tau_{1,0}) = \{\{U, D\}\}$ and $\mathcal{S}_2(\tau_{2,0}) = \{\{L, L'\}, \{L, L', R\}\}$. Thus, by Theorem 1, no action can be selected for $\tau_{1,0}$.

Example 2 *Modifying Morris, Takahashi, and Tercieux (2012, Example 2), consider the following*

[22] In Penta (2013, Section 3.2), he observes that his Theorem 1 remains true if $\mathcal{A}^\infty$ is replaced by the set of actions for which there exists payoff states that make these actions uniquely rationalizable. This observation follows from Frankel, Morris, and Pauzner (2003) and can be applied to our auction example but not to the Cournot example. In Penta (2013, Section 4.4), he further considers replacing $\mathcal{A}^\infty$ with $\mathcal{A}_i^* \subset R_i^{\mathrm{u}}$ such that each $a_i \in \mathcal{A}_i^*$ is a unique best reply to some belief over $\Theta \times \mathcal{A}_{-i}^*$. He then claims in Proposition 3 that every action in $ICR_i(t_i; \mathcal{A}^*)$ can be selected for t_i. Again, the set $ICR_i(t_i; \mathcal{A}^*)$ should be defined recursively as we do in Corollary 1 in order to make his Proposition 3 correct (Example 1). This modified version is still not necessary for the selections (Example 2).

extensive-form game



and its reduced normal form

θ :	L	R	X
U	$1,3$	$0,2$	$1,1$
Du	$\theta,2$	$1,3$	$2,1$
Dd	$\theta,2$	$1,3$	$0,1$

with $\theta \in \Theta = \{0,2\}$. *(The following argument is insensitive to small payoff perturbations on terminal nodes in the extensive form.) Let* $\tau_{i,0}$ *be the type of player* i *with complete information about* $\theta = 0$. *Then we have* $\mathcal{R}_1^{\uparrow} = \mathcal{S}_1(\tau_{1,0}) = \{\{Du, Dd\}, \{U, Du, Dd\}\}$ *and* $\mathcal{R}_2^{\uparrow} = \mathcal{S}_2(\tau_{2,0}) = \{\{R\}, \{L, R\}, \{R, X\}, \{L, R, X\}\}$. *Thus, by Theorem 1,* R *can be selected for* $\tau_{2,0}$. *On the other hand, we have* $R_1^{\mathrm{u}} = \varnothing$ *and* $R_2^{\mathrm{u}} = \{R\}$, *and hence* $S_1^{\mathrm{u}}(\tau_{1,0}) = S_2^{\mathrm{u}}(\tau_{2,0}) = \varnothing$. *The example shows that while* $a_i \in S_i^{\mathrm{u}}(t_i)$ *is a sufficient condition for* a_i *to be selected for* t_i, *it is not necessary and misses cases where some selection is possible. Note that the state space is too small to satisfy the extensive-form richness condition in Chen (2012).*

4.4 The Structure Theorem and Generic Uniqueness

Based upon our characterization of the selections, this subsection fully characterizes the structure theorem as well as generic uniqueness. Unlike the existing papers, our characterizations will be stated in terms of the primitives, and will not presuppose the existence of dominant actions or any richness condition. The characterizations will thus delineate an exact boundary of the WY critique.

Recall that R_i^{u} is the set of actions that are uniquely rationalizable for some type. We first characterize the structure theorem.

Corollary 2 *The following three conditions are equivalent:*

1. *for any type $t_i \in T_i^*$, any action in $ICR_i(t_i)$ can be selected for t_i;*
2. *for any finite type $t_i \in T_i^*$, any action in $ICR_i(t_i)$ can be selected for t_i;*
3. *for any $i \in I$ and $R_i \in \mathcal{R}_i^{\downarrow}$, we have $R_i \subset R_i^{\mathrm{u}}$.*

Proof "1 $\Rightarrow$ 2" is obvious, and "2 $\Rightarrow$ 1" follows from Lemma 1.

For "2 $\Rightarrow$ 3" for any $i \in I$ and $R_i \in \mathcal{R}_i^{\downarrow}$, by Lemma 1, there exists a finite type t_i such that $R_i \subset ICR_i(t_i)$. Thus we have $R_i \subset ICR_i(t_i) \subset R_i^{\mathrm{u}}$.

For "3 $\Rightarrow$ 2" by Corollary 1, it suffices to show that $ICR_i(t_i) \subset S_i^{\mathrm{u}}(t_i)$ for any finite type t_i. We fix any finite model (T, κ), and prove by induction that $ICR_i(t_i) \subset S_i^{\mathrm{u},n}(t_i)$ for any $i \in I$ and $t_i \in T_i$. The case of $n = 0$ is obvious. Now suppose that $ICR_i(t_i) \subset S_i^{\mathrm{u},n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Given any $i \in I$ and $t_i \in T_i$, consider any valid conjecture $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ for t_i. Then, $\mu_i^{\mathrm{u}} = \nu_i$ satisfies (i) and (ii) in (6) because ν_i is valid for t_i and we assume the induction hypothesis. Thus $BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i\right) \subset S_i^{\mathrm{u},n}(t_i)$. By (1), we have $ICR_i(t_i) \subset S_i^{\mathrm{u},n}(t_i)$. ■

This corollary reproduces Weinstein and Yildiz (2007b, Proposition 1) and Chen (2012, Theorem 1). In words, a necessary and sufficient condition for every rationalizable action to be selected for any (finite) type (i.e., the structure theorem) is that every rationalizable action is uniquely rationalizable for some type. This condition is called richness in uniquely rationalizable actions (RURA) in Chen (2012). Note that the RURA condition is not imposed on the primitives directly, but our algorithms to compute $\mathcal{R}_i^{\uparrow}$ and $\mathcal{R}_i^{\downarrow}$ provide a way to decide whether the RURA condition holds from the primitives.

We then turn to characterize generic uniqueness.

Corollary 3 *The following two conditions are equivalent:*

1. *for any $i \in I$, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is open and dense in T_i^*;*

2. *for any $i \in I$ and $R_i \in \mathcal{R}_i^{\uparrow}$, we have $R_i \cap R_i^{\mathrm{u}} \neq \varnothing$.*

Proof For the "1 $\Rightarrow$ 2" direction, for any $i \in I$ and $R_i \in \mathcal{R}_i^{\uparrow}$, by Lemma 1, there exists a finite type t_i such that $R_i \supset ICR_i(t_i)$. Since $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is dense in T_i^* and $ICR_i(\cdot)$ is upper hemicontinuous, we have $ICR_i(t_i) \cap R_i^{\mathrm{u}} \neq \varnothing$, and hence $R_i \cap R_i^{\mathrm{u}} \neq \varnothing$.

For the "2 $\Rightarrow$ 1" direction, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is open in T_i^* since $ICR_i(\cdot)$ is upper hemicontinuous. To show that $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is dense in T_i^*, by Lemma 1 and Corollary 1, it suffices to show that $ICR_i(t_i) \cap S_i^{\mathrm{u}}(t_i) \neq \varnothing$ for any finite type t_i. We fix any finite model (T, κ), and prove by induction that $ICR_i(t_i) \cap S_i^{\mathrm{u},n}(t_i) \neq \varnothing$ for any $i \in I$ and $t_i \in T_i$. The case of $n = 0$ is obvious. Now suppose that $ICR_i(t_i) \cap S_i^{\mathrm{u},n-1}(t_i) \neq \varnothing$ for any $i \in I$ and $t_i \in T_i$. Then, given any $i \in I$ and $t_i \in T_i$, there exists $\nu_i \in \Delta(\Theta \times T_{-i} \times A_{-i})$ such that $\mathrm{marg}_{\Theta \times T_{-i}} \nu_i = \kappa_{t_i}$ and $\nu_i \left[a_{-i} \in ICR_{-i}(t_{-i}) \cap S_{-i}^{\mathrm{u},n-1}(t_{-i})\right] = 1$. Since ν_i is a valid conjecture for t_i and $\mu_i^{\mathrm{u}} = \nu_i$ satisfies (i) and (ii) in (6), by (1), we have $ICR_i(t_i) \cap S_i^{\mathrm{u},n}(t_i) \supset BR_i\left(\mathrm{marg}_{\Theta \times A_{-i}} \nu_i\right) \neq \varnothing$. ■

Corollary 3 shows that a necessary and sufficient condition for types with uniquely rationalizable actions to be generic in the universal type space (i.e., generic uniqueness) is that every set in the upper ICR collection contains some action that can be identified with a singleton in the upper ICR collection, i.e., every minimal ICR set is a singleton. Note that generic uniqueness (Condition 1 in Corollary 3) is a weaker statement than the structure theorem (Conditions 1 and 2 in Corollary 2): the former requires that *some* rationalizable action be selected for each type, whereas the latter only requires that *every* rationalizable action be selected for each type. In particular, Corollary 3 can apply to games with weakly dominated actions in any state (see the next Subsection). In WY, the richness condition implies both results and renders their distinction moot.

4.5 Applications

4.5.1 The Cournot Example Revisited

So far we assume that game G is finite in our formal analysis, but the Cournot example in Subsection 3.1 has infinitely many actions. There are two ways to apply our results to this example. One is to discretize the action space as specified at the end of Subsection 3.1. The other is to analyze the infinite game directly. To do so, observe that the proof of $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ in Proposition 3 and also the proof of Corollaries 1 and 2 do not depend on the finiteness assumption of A_i. Thus, it follows from Proposition 1 and Corollary 2 that (a) if $\theta_H/\theta_L > (|I|-1)/2$, we can select every $q \in [0, \theta_H/2]$ for every type t_i, whereas (b) if $\theta_H/\theta_L \leq (|I|-1)/2$, no type has a uniquely rationalizable action. Therefore, the sharp discontinuity between the two cases remains regarding the selections and the structure theorem.

Note that for $\theta_H/\theta_L > (|I|-1)/2$, this structure theorem without discretization is slightly different from WY's original one that requires the openness of the set of types for which a given action is uniquely rationalizable (Weinstein and Yildiz, 2007b, p. 372). Indeed, when the action set is infinite, even though the ICR correspondence remains to be upper hemicontinuous (Weinstein and Yildiz, 2012, Proposition 3), $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ need not be open. Nonetheless, $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is still a countable intersection of $\{t_i \in T_i^* : \text{diameter of } ICR_i(t_i) < 1/n\}$, each of which is open (because $ICR(\cdot)$ is upper hemicontinuous) and dense (because every $q \in [0, \theta_H/2]$ can be selected for every type t_i). Therefore, the generic uniqueness holds in a slightly weaker sense, i.e., $\{t_i \in T_i^* : |ICR_i(t_i)| = 1\}$ is a residual set in T_i^*.

4.5.2 The Auction Example Revisited

Recall the auction example in Subsection 3.2. It is straightforward to verify that $\{0\} \in \mathcal{R}_i^{\uparrow,1}$, and inductively, for any $1 \leq n \leq 10$, $\{k\} \in \mathcal{R}_i^{\uparrow,n}$ for $k = 0, \ldots, n-1$. Moreover, we have $\{10\} \notin \mathcal{R}_i^{\uparrow,n}$ since 0 is also a best reply whenever 10 is a best reply. It then follows from Proposition 2 that $R_i^{\mathrm{u}} = \{0, 1, \ldots, 9\}$.

We can also verify that $\{10\} \in \mathcal{R}_i^{\downarrow}$ by considering the belief that all bidders have values 10 and bid 10. Thus, $\{10\} \in \mathcal{R}_i^{\downarrow}$ and $10 \notin R_i^{\mathrm{u}}$. It follows from Corollary 2 that the structure theorem does not hold in this example. Next, for any minimal $R_i \in \mathcal{R}_i^{\uparrow}$, if $10 \in R_i$, we also have $0 \in R_i$. Thus, $R_i \cap R_i^{\mathrm{u}} \neq \varnothing$ for every $R_i \in \mathcal{R}_i^{\uparrow}$.[23] It follows from Corollary 3 that the generic uniqueness holds in this example.

Finally, recall that $\{0, 1, \ldots, 9\}$ is a robust prediction for the type $\tau_{i,10}$ with complete information that all bidders have values 10. We now show that $\{9\}$ is the sharpest robust prediction for $\tau_{i,10}$ and hence $\{9\}$ is a strongly robust prediction, which is in contrast with $ICR_i(\tau_{i,10}) = \{0, 1, \ldots, 10\}$. To see this, we show that $\{9\}$ is the only minimal set in $\mathcal{S}_i(\tau_{i,10})$.

We show first that 0 does not belong to any minimal set in $\mathcal{S}_i^1(\tau_{i,10})$. Indeed, since $R_i^{\mathrm{u}} = \{0, 1, \ldots, 9\}$, the minimal sets in $\mathcal{R}_i^{\uparrow}$ are $\{0\}, \{1\}, \ldots, \{9\}$. Thus, to determine the minimal sets in $\mathcal{S}_i^1(\tau_{i,10})$, it is without loss of generality to consider beliefs that concentrates on $\{0\}, \{1\}, \ldots, \{9\}$. In this case, 0 is never a best reply against any belief. Indeed, if a belief assigns a positive probability that all the opponents bid 0, then bidding 1 is strictly better than bidding 0 (since $10/|I| < 9$); if a belief assigns probability zero that all the opponents bid 0, then bidding 9 to win with a positive probability is strictly better than bidding 0. Hence, 0 does not belong to any minimal set in $\mathcal{S}_i^1(\tau_{i,10})$. Moreover, each $\{b\}$ with $b \in \{1, \ldots, 9\}$ is a minimal set in $\mathcal{S}_i^1(\tau_{i,10})$ since b is the unique best reply against a belief concentrating on $b-1$ (since $0 < (10-b)/|I| < 10-b-1$).

Inductively, we can show that for any n, $b \leq \min(n-1, 8)$ (as well as $b = 10$) does not belong to any minimal set in $\mathcal{S}_i^n(\tau_{i,10})$. Finally, since bidding 9 is a strict best reply against a belief concentrating on $\{9\}$, we have $\{9\} \in \mathcal{S}_i^n(\tau_{i,10})$ for every n. Therefore, $\{9\}$ is the only minimal set in $\mathcal{S}_i(\tau_{i,10})$.

[23] In fact, it can be verified that $\mathcal{R}_i^{\uparrow} = \mathcal{A}_i \backslash \{\{10\}\}$ and $\mathcal{R}_i^{\downarrow} = \mathcal{A}_i$.

5 Infinite Types

This section extends our results to infinite types. The key to such an extension is a measurability requirement. To see this, suppose instead that we adopt the same definition of $\mathcal{S}_i^n(t_i)$ as in (6) for finite types. Since $\mathcal{S}_i^n(t_i)$ is specified on a type-by-type basis, we may not be able to find (μ_i, μ_i') that depends on (t_i, R_i) measurably, which is an indispensable step in the proof of Proposition 3. To circumvent this problem, we introduce a fixed-point counterpart of $\mathcal{S}_i^n(t_i)$ that already incorporates the measurability of (μ_i, μ_i') as a part of definition.

Formally, fix any (possibly infinite) model (T, κ). A profile $\left(\widetilde{\mathcal{S}}_i\right)_{i\in I}$ of measurable mappings $\widetilde{\mathcal{S}}_i \colon T_i \to 2^{\mathcal{R}_i^{\uparrow}} \setminus \{\varnothing\}$ is called an $\mathcal{R}^{\uparrow}$*-perturbed curb collection* on (T, κ) if for every $i \in I$ and $\varepsilon \in (0, 1]$, there exists a measurable mapping

$$(\mu, \mu') \colon T_i \times \mathcal{R}_i^{\uparrow} \to \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$$

such that for each $t_i \in T_i$ and $R_i \in \widetilde{\mathcal{S}}_i(t_i)$,

(i) $\operatorname{marg}_{\Theta\times T_{-i}} \mu_{t_i,R_i} = \kappa_{t_i}$;

(ii) $\mu_{t_i,R_i}\left[\left\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \widetilde{\mathcal{S}}_{i-i}(t_{-i})\right\}\right] = 1$;

(iii) $R_i \supset \bigcup_{(\pi_i,\pi_i')\in \Pi_i^{\mu_{t_i,R_i}} \times \Pi_i^{\mu'_{t_i,R_i}}} BR_i\left((1-\varepsilon)\pi_i + \varepsilon\pi_i'\right)$,

where $\Pi_i^{\mu_{t_i,R_i}}$ and $\Pi_i^{\mu'_{t_i,R_i}}$ are the sets of π_i satisfying (2)-(3) and (4)-(5), respectively, with the additional measurability requirement on φ_i. Note that $\mathcal{R}^{\uparrow}$-perturbed curb collections are defined on each model (T, κ), which may be infinite, but much smaller than the universal model (T^*, κ^*).

The following is a generalization of Proposition 3 to infinite types. The proof is in Appendix A.4.

Proposition 4 *For any model (T, κ), $(\mathcal{S}_i^*|_{T_i})_{i\in I}$ is the largest $\mathcal{R}^{\uparrow}$-perturbed curb collection on (T, κ).*

By Proposition 4, we can characterize all selections and robust predictions in terms of $\mathcal{R}^{\uparrow}$-perturbed curb collections.

Theorem 3 *Fix a model (T, κ). Action a_i can be selected for type $t_i \in T_i$ if and only if $\{a_i\} \in \widetilde{\mathcal{S}}_i(t_i)$ for some $\mathcal{R}^{\uparrow}$-perturbed curb collection $\left(\widetilde{\mathcal{S}}_j\right)_{j \in I}$ on (T, κ).*

Theorem 4 *Fix a model (T, κ). Prediction P_i is robust (resp. strongly robust) for type $t_i \in T_i$ if and only if $P_i \cap R_i \neq \varnothing$ (resp. $P_i \subset R_i$) for any $\mathcal{R}^{\uparrow}$-perturbed curb collection $\left(\widetilde{\mathcal{S}}_j\right)_{j \in I}$ on (T, κ) and any $R_i \in \widetilde{\mathcal{S}}_i(t_i)$.*

6 Conclusion

In this paper, without imposing any structure on payoffs, we have characterized all selections from and all robust predictions of rationalizable actions for any finite type. It is worth noting that we achieve the characterization by utilizing a novel approach, namely the collection-based approach first proposed in Chen, Takahashi, and Xiong (2014). More precisely, we study *collections of subsets of actions* and their best reply property, compared to the previous literature that primarily focuses on the best reply property of *subsets of actions*. We believe that this collection-based approach is useful in investigating other related questions as well.

A Appendix

A.1 Proof of Proposition 1

Proof of Proposition 1 (a) The "only if" direction is obvious. To show the "if" direction, let $r = (|I| - 1)/2 \geq 1$ and $x = (\theta_H - r\theta_L)/2 > 0$.

Claim 1 *For any $m \geq 1$, there exist $\tau_{i,m,L}, \tau_{i,m,H} \in T_i^*$ such that*

$$ICR_i(\tau_{i,1,L}) \subset \left[0, \frac{\theta_L}{2}\right],$$
$$ICR_i(\tau_{i,m,L}) \subset \left[0, \max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right)\right], \forall m \geq 2$$
$$ICR_i(\tau_{i,m,H}) \subset \left[\min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-2})x\right), \frac{\theta_H}{2}\right], \forall m \geq 1.$$

Proof of Claim 1 We construct desired types inductively. For $m = 1$, we can take type $\tau_{i,1,L}$ to be any type whose first-order belief puts probability 1 on $\theta = \theta_L$. Then we have $ICR_i(\tau_{i,1,L}) \subset [0, \theta_L/2]$.

For any $m \geq 1$, we take type $\tau_{i,m,H}$ to be the type who puts probability 1 on $\theta = \theta_H$ and $t_j = \tau_{j,m,L}$ for any $j \neq i$. By (1) and the induction hypothesis, any action that is rationalizable for $\tau_{i,m,H}$ is bounded from below by

$$\frac{1}{2}\left(\theta_H - (|I| - 1)\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right)\right) = \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-2})x\right).$$

Thus we have $ICR_i(\tau_{i,m,H}) \subset \left[\min\left(\theta_H/2, (1 + r^2 + \cdots + r^{2m-2})x\right), \theta_H/2\right]$.

Similarly, for any $m \geq 2$, we take type $\tau_{i,m,L}$ to be the type who puts probability 1 on $\theta = \theta_L$ and $t_j = \tau_{j,m-1,H}$ for any $j \neq i$. By (1) and the induction hypothesis, any action that is rationalizable for $\tau_{i,m,L}$ is bounded from above by

$$\max\left(0, \frac{1}{2}\left(\theta_L - (|I| - 1)\min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2m-4})x\right)\right)\right)$$
$$= \max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2m-3})x\right).$$

Thus we have $ICR_i(\tau_{i,m,L}) \subset \left[0, \max\left(0, \theta_L/2 - (r + r^3 + \cdots + r^{2m-3})x\right)\right]$. ■

Claim 2 *For any $n \geq 0$ and any*

$$q \in \left[0, \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2n})x\right)\right] \cup \left[\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2n+1})x\right), \frac{\theta_H}{2}\right],$$

there exists $\tau_{i,q} \in T_i^*$ such that $ICR_i(\tau_{i,q}) = \{q\}$.

Proof of Claim 2 We construct desired types inductively. For $n = 0$, we take $\tau_{i,0} = \tau_{i,m,L}$ in Claim 1 with sufficiently large m. Then we have $ICR_i(\tau_{i,0}) = \{0\}$.

Also, for $n = 0$ and any $q \in [\theta_L/2, \theta_H/2]$, we take $\tau_{i,q}$ to be the type who puts probability $(2q - \theta_L)/(\theta_H - \theta_L)$ on $\theta = \theta_H$ and $t_j = \tau_{j,0}$ for any $j \neq i$, and probability $(\theta_H - 2q)/(\theta_H - \theta_L)$ on $\theta = \theta_L$ and $t_j = \tau_{j,0}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$.

For any $n \geq 1$ and any $q \in [0, \min(\theta_H/2, (1 + r^2 + \cdots + r^{2n})x)]$, let $q' = (\theta_H - 2q)/(|I| - 1)$. Since

$$\begin{aligned} q' &\in \left[\frac{\theta_H - 2\min\left(\theta_H/2, (1 + r^2 + \cdots + r^{2n})x\right)}{|I| - 1}, \frac{\theta_H}{|I| - 1}\right] \\ &\subset \left[\max\left(0, \frac{\theta_L}{2} - (r + r^3 + \cdots + r^{2n+1})x\right), \frac{\theta_H}{2}\right] \end{aligned}$$

by the induction hypothesis, there exists $\tau_{j,q'} \in T_i^*$ such that $ICR_i(\tau_{i,q'}) = \{q'\}$. We take $\tau_{i,q}$ to be the type who puts probability 1 on $\theta = \theta_H$ and $t_j = \tau_{j,q'}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$.

Similarly, for any $n \geq 1$ and any $q \in [\max\left(0, \theta_L/2 - (r + r^3 + \cdots + r^{2n+1})x\right), \theta_H/2]$, if $q \geq \theta_L/2$, then the desired $\tau_{i,q}$ is already constructed in the case of $n = 0$. If $q < \theta_L/2$, then let $q'' = (\theta_L - 2q)/(|I| - 1)$. Since

$$\begin{aligned} q'' &\in \left(0, \frac{\theta_L - 2\max\left(0, \theta_L/2 - (r + r^3 + \cdots + r^{2n+1})x\right)}{|I| - 1}\right] \\ &\subset \left[0, \min\left(\frac{\theta_H}{2}, (1 + r^2 + \cdots + r^{2n})x\right)\right] \end{aligned}$$

by the induction hypothesis, there exists $\tau_{j,q''} \in T_i^*$ such that $ICR_i(\tau_{i,q''}) = \{q''\}$. We take $\tau_{i,q}$ to be the type who puts probability 1 on $\theta = \theta_L$ and $t_j = \tau_{j,q''}$ for any $j \neq i$. By (1), we have $ICR_i(\tau_{i,q}) = \{q\}$. ■

By taking $n \to \infty$ in Claim 2, we can construct $\tau_{i,q} \in T_i^*$ for any $q \in [0, \theta_H/2]$.

(b) For each $t_i \in T_i^*$, we have $ICR_i^1(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$. For each $t_i \in T_i^*$ and $q \in \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$, let $q(t_i) = \left(\mathbb{E}_{t_i^1}(\theta) - 2q\right)/(|I| - 1)$. Then q is a best response to the conjecture ν_i such that $\text{marg}_{\Theta\times T^*_{-i}}\nu_i = \kappa^*_{t_i}$ and $\nu_i\left[a_{-i} = q(t_i)\right] = 1$. Also,

$$q(t_i) \in \left[0, \frac{\mathbb{E}_{t_i^1}(\theta)}{|I|-1}\right] \subset \left[0, \frac{\theta_H}{|I|-1}\right] \subset \left[0, \frac{\theta_L}{2}\right] \subset \left[0, \frac{\mathbb{E}_{t_i^1}(\theta)}{2}\right]$$

for any $t_{-i} \in T^*_i$. Thus we have $ICR_i(t_i) = \left[0, \mathbb{E}_{t_i^1}(\theta)/2\right]$. ■

A.2 Proof of Proposition 2

We first prove the following lemma.

Lemma 2 *For any $n \geq 0$, we have (a) $\mathcal{R}_i^{\uparrow,n} = \{R_i \in \mathcal{A}_i : \exists t_i \in T^*_i \text{ s.t. } R_i \supset ICR_i^n(t_i)\}$; (b) $\mathcal{R}_i^{\downarrow,n} = \{R_i \in \mathcal{A}_i : \exists t_i \in T^*_i \text{ s.t. } R_i \subset ICR_i^n(t_i)\}$.*

Proof The proof of (b) is similar to the proof of (a) and thus omitted. We prove (a) by induction. The case for $n = 0$ is obvious. Suppose that the claim holds for $n-1$ and we prove the case for n.

For "$\supset$", suppose that $R_i \supset ICR_i^n(t_i)$ for some $t_i \in T^*_i$. Define $\mu_i \in \Delta(\Theta \times \mathcal{A}_{-i})$ such that

$$\mu_i[\theta, R_{-i}] = \kappa^*_{t_i}\left[\left\{(\theta, t_{-i}) : ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}\right] \tag{9}$$

for every $(\theta, R_{-i}) \in \Theta \times \mathcal{A}_{-i}$.[24] By the induction hypothesis, $\mu_i \in \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow,n-1}\right)$. We prove that $R_i \supset BR_i(\pi_i)$ for every $\pi_i \in \Pi_i^{\mu_i}$ to conclude $R_i \in \mathcal{R}_i^{\uparrow,n}$. Pick any $\pi_i \in \Pi_i^{\mu_i}$. Then there exists a function $\varphi_i \colon \Theta \times \mathcal{A}_{-i} \to \Delta(A_{-i})$ such that (2) and (3) hold. Define $\nu_i \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ such that

$$\begin{aligned}
&\nu_i\left[\{\theta\} \times E_{-i} \times \{a_{-i}\}\right] \\
= &\sum_{R_{-i}\in\mathcal{A}_{-i}} \kappa^*_{t_i}\left[\left\{(\theta, t_{-i}) : t_{-i} \in E_{-i} \text{ and } ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}\right] \varphi_i(\theta, R_{-i})[a_{-i}]
\end{aligned} \tag{10}$$

[24] By Dekel, Fudenberg, and Morris (2007, Lemma 1), $ICR_j^{n-1}(\cdot)$ is upper hemicontinuous when T^*_j is endowed with the product topology. Thus, $\left\{(\theta, t_{-i}) : ICR_{-i}^{n-1}(t_{-i}) = R_{-i}\right\}$ is measurable.

for every measurable $E_{-i} \subset T^*_{-i}$ and $(\theta, a_{-i}) \in \Theta \times A_{-i}$. It then follows that $\operatorname{marg}_{\Theta \times T_{-i}} \nu_i = \kappa^*_{t_i}$; $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i}) \right] = 1$ by (2); $\operatorname{marg}_{\Theta \times A_{-i}} \nu_i = \pi_i$ because

$$\begin{aligned}
\operatorname{marg}_{\Theta \times A_{-i}} \nu_i[\theta, a_{-i}] &= \sum_{R_{-i} \in \mathcal{A}_{-i}} \kappa^*_{t_i} \left[\left\{ (\theta, t_{-i}) : ICR^{n-1}_{-i}(t_{-i}) = R_{-i} \right\} \right] \varphi_i(\theta, R_{-i})[a_{-i}] \\
&= \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i[\theta, R_{-i}] \, \varphi_i(\theta, R_{-i})[a_{-i}] \\
&= \pi_i[\theta, a_{-i}]
\end{aligned}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$, where the three equalities follow from (10), (9), and (3), respectively. Thus, we have $R_i \supset ICR^n_i(t_i) \supset BR_i(\pi_i)$.

For "$\subset$", suppose that $R_i \in \mathcal{R}^{\uparrow,n}_i$. Then, there exists $\mu_i \in \Delta(\Theta \times \mathcal{R}^{\uparrow,n-1}_{-i})$ such that $R_i \supset BR_i(\pi_i)$ for every $\pi_i \in \Pi^{\mu_i}_i$. By the induction hypothesis, for every $R_{-i} \in \mathcal{R}^{\uparrow,n-1}_{-i}$, there exists $\tau_{-i,R_{-i}} \in T^*_{-i}$ such that $R_{-i} \supset ICR^{n-1}_{-i}(\tau_{-i,R_{-i}})$. Define $t_i \in T^*_i$ with $\kappa^*_{t_i}$ having a finite support such that

$$\kappa^*_{t_i}[\theta, t_{-i}] = \mu_i\left[\left\{ (\theta, R_{-i}) : \tau_{-i,R_{-i}} = t_{-i} \right\} \right].$$

We now show $R_i \supset ICR^n_i(t_i)$. Pick any conjecture $\nu_i \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ such that $\operatorname{marg}_{\Theta \times T^*_{-i}} \nu_i = \kappa^*_{t_i}$ and $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i}) \right] = 1$. Let $\pi_i = \operatorname{marg}_{\Theta \times A_{-i}} \nu_i$. Define φ_i as the conditional probability of ν_i on each $\left(\theta, \tau_{-i,R_{-i}}\right)$, i.e., $\varphi_i(\theta, R_{-i})[a_{-i}] = \nu_i\left[a_{-i} \mid \theta, \tau_{-i,R_{-i}} \right]$. (If $\kappa^*_{t_i}\left[\theta, \tau_{-i,R_{-i}}\right] = 0$, then pick $\varphi_i(\theta, R_{-i}) \in \Delta(R_{-i})$ arbitrarily.) Then, (2) holds because $\nu_i \left[a_{-i} \in ICR^{n-1}_{-i}(t_{-i}) \right] = 1$ and $R_{-i} \supset ICR^{n-1}_{-i}\left(\tau_{-i,R_{-i}}\right)$ for every $R_{-i} \in \mathcal{R}^{\uparrow,n-1}_{-i}$; (3) holds because

$$\begin{aligned}
\pi_i[\theta, a_{-i}] &= \operatorname{marg}_{\Theta \times A_{-i}} \nu_i[\theta, a_{-i}] \\
&= \sum_{t_{-i} \in T^*_{-i}} \kappa^*_{t_i}[\theta, t_{-i}] \nu_i[a_{-i} \mid \theta, t_{-i}] \\
&= \sum_{t_{-i} \in T^*_{-i}} \sum_{R_{-i} \in \mathcal{A}_{-i} : \tau_{-i,R_{-i}} = t_{-i}} \mu_i[\theta, R_{-i}] \varphi(\theta, R_{-i})[a_{-i}] \\
&= \sum_{R_{-i} \in \mathcal{A}_{-i}} \mu_i[\theta, R_{-i}] \varphi(\theta, R_{-i})[a_{-i}]
\end{aligned}$$

for every $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Thus, we have $\pi_i \in \Pi^{\mu_i}_i$, and hence $R_i \supset BR_i(\pi_i)$. Therefore, we have $R_i \supset ICR^n_i(t_i)$. ∎

We now turn to prove Proposition 2.

Proof of Proposition 2 (a) For "$\subset$", suppose that $R_i \in \mathcal{R}_i^{\uparrow,n}$ for some n. By Lemma 2(a), there exists $t_i \in T_i^*$ such that $R_i \supset ICR_i^n(t_i)$. Since $R_i \supset ICR_i^n(t_i) \supset ICR_i(t_i)$, we have $R_i \in \mathcal{R}_i^{\uparrow}$.

For "$\supset$", suppose that $R_i \in \mathcal{R}_i^{\uparrow}$. Then there exist $t_i \in T_i^*$ and m such that $R_i \supset ICR_i(t_i) = ICR_i^m(t_i)$. By Lemma 2(a), we have $R_i \in \mathcal{R}_i^{\uparrow,m} \subset \mathcal{R}_i^{\uparrow,n}$ for any $n \geq \sum_i 2^{|A_i|} - 2|I|$.

(b) For "$\subset$", suppose that $R_i \in \mathcal{R}_i^{\downarrow,n}$ for some $n \geq \sum_i 2^{|A_i|} - 2|I|$. For each m, since $R_i \in \mathcal{R}_i^{\downarrow,n} \subset \mathcal{R}_i^{\downarrow,m}$, by Lemma 2(b), there exists $t_{i,m} \in T_i^*$ such that $R_i \subset ICR_i^m(t_{i,m})$. Since T_i^* is a compact metric space, $\{t_{i,m}\}$ admits a convergent subsequence $\{t_{i,m_k}\}$. We denote its limit by t_i. For any m and $m_k \geq m$, we have $R_i \subset ICR_i^{m_k}(t_{i,m_k}) \subset ICR_i^m(t_{i,m_k})$. Since $t_{i,m_k} \to t_i$ as $k \to \infty$ and $ICR_i^m(\cdot)$ is upper hemicontinuous, we have $R_i \subset ICR_i^m(t_i)$. Since m is arbitrary, we have $R_i \subset ICR_i(t_i)$, and hence $R_i \in \mathcal{R}_i^{\downarrow}$.

For "$\supset$", suppose that $R_i \in \mathcal{R}_i^{\downarrow}$. Then there exists $t_i \in T_i^*$ such that $R_i \subset ICR_i(t_i) \subset ICR_i^n(t_i)$ for any n. By Lemma 2(b), we have $R_i \in \mathcal{R}_i^{\downarrow,n}$. ■

A.3 Proof of Proposition 3

We prove Proposition 3 in the following two lemmas.

Lemma 3 *$\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ for finite type t_i.*

Proof Define $\mathcal{S}_i^{*,0}(t_i) := \mathcal{R}_i^{\uparrow}$ and

$$\mathcal{S}_i^{*,n}(t_i) := \left\{R_i \in \mathcal{A}_i : \exists \{t_{i,m}\}_{m=0}^{\infty} \subset T_i^* \text{ s.t. } t_{i,m}^n \to t_i^n \text{ as } m \to \infty \text{ and } R_i \supset ICR_i(t_{i,m}), \forall m\right\}$$

for each $n \geq 1$. We show that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$, and thus $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^*(t_i)$ by taking a diagonal sequence. We fix a finite model (T, κ), and prove by induction that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$. For $n = 0$, we have $\mathcal{S}_i(t_i) \subset \mathcal{R}_i^{\uparrow} = \mathcal{S}_i^{*,0}(t_i)$. Suppose that $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n-1}(t_i)$ for any $i \in I$ and $t_i \in T_i$, and we prove $\mathcal{S}_i(t_i) \subset \mathcal{S}_i^{*,n}(t_i)$ for any $i \in I$ and $t_i \in T_i$. Let $i \in I$, $t_i \in T_i$, and $R_i \in \mathcal{S}_i(t_i)$. By the fixed-point property of $\mathcal{S}_i(\cdot)$, for each m, there exists $(\mu_{i,m}, \mu'_{i,m}) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that (i)-(iii) in (7) with $\varepsilon = \frac{1}{m+1}$

holds. First, for each $R_{-i} \in \mathcal{R}^{\uparrow}$, there exists $\tau_{-i,R_{-i}} \in T^*_{-i}$ such that $R_{-i} \supset ICR_{-i}\left(\tau_{-i,R_{-i}}\right)$. Also, for each $t_{-i} \in T_{-i}$ and $R_{-i} \in \mathcal{S}_{-i}\left(t_{-i}\right)$, by the induction hypothesis, there is some sequence of types $\left\{\tau_{t_{-i},R_{-i},m}\right\}_{m=0}^{\infty} \subset T^*_{-i}$ such that $\tau^{n-1}_{t_{-i},R_{-i},m} \to t^{n-1}_{-i}$ as $m \to \infty$ and $R_{-i} \supset ICR_{-i}\left(\tau_{t_{-i},R_{-i},m}\right)$ for every m. (If $n=1$, we set $\tau_{t_{-i},R_{-i},m} = \tau_{-i,R_{-i}}$.) Define $t_{i,m} \in T^*_i$ with $\kappa^*_{t_{i,m}}$ having a finite support such that

$$\begin{aligned}\kappa^*_{t_{i,m}}\left[\theta, s_{-i}\right] = \frac{m}{m+1}&\mu_{i,m}\left[\left\{\left(\theta, t_{-i}, R_{-i}\right) : \tau_{t_{-i},R_{-i},m} = s_{-i}\right\}\right] \\ &+ \frac{1}{m+1}\mu'_{i,m}\left[\left\{\left(\theta, R_{-i}\right) : \tau_{-i,R_{-i}} = s_{-i}\right\}\right] \end{aligned} \tag{11}$$

for every $\left(\theta, s_{-i}\right) \in \Theta \times T^*_{-i}$. Since $\tau^{n-1}_{t_{-i},R_{-i},m} \to t^{n-1}_{-i}$ as $m \to \infty$ and $\operatorname{marg}_{\Theta \times T_{-i}} \mu_{i,m} = \kappa_{t_i}$ for every m, it follows that $t^n_{i,m} \to t^n_i$ as $m \to \infty$.

Finally, we show that $R_i \supset ICR_i\left(t_{i,m}\right)$ for every m. Pick any $a_i \in ICR_i\left(t_{i,m}\right)$ and we show $a_i \in R_i$. Since $a_i \in ICR_i\left(t_{i,m}\right)$, by (1), there is a valid conjecture $\nu_{i,m} \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ for $t_{i,m}$ such that $a_i \in BR_i\left(\operatorname{marg}_{\Theta \times A_{-i}} \nu_{i,m}\right)$. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. For each $\left(\theta, R_{-i}\right) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, let $\psi^{\alpha_i}_{-i}(\theta, R_{-i}) \in R_{-i}$ be one of the action profiles of player i's opponents that favor action a_i most relative to α_i, i.e.,

$$\psi^{\alpha_i}_{-i}(\theta, R_{-i}) \in \underset{a_{-i} \in R_{-i}}{\arg\max}\left[u_i(\theta, a_i, a_{-i}) - u_i(\theta, \alpha_i, a_{-i})\right]. \tag{12}$$

Since $\operatorname{marg}_{\Theta \times T^*_{-i}} \nu_{i,m} = \kappa^*_{t_{i,m}}$, $\nu_{i,m}\left[a_{-i} \in ICR_{-i}\left(t_{-i}\right)\right] = 1$, and $a_i \in BR_i\left(\operatorname{marg}_{\Theta \times A_{-i}} \nu_{i,m}\right)$, it follows that a_i is no worse than α_i against $\pi^*_{i,m}$, where

$$\pi^*_{i,m}\left[\theta, a_{-i}\right] = \kappa^*_{t_{i,m}}\left[\left\{\left(\theta, s_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(s_{-i})) = a_{-i}\right\}\right] \tag{13}$$

for every $\left(\theta, a_{-i}\right) \in \Theta \times A_{-i}$. Let

$$\pi_{i,m}\left[\theta, a_{-i}\right] = \mu_{i,m}\left[\left\{\left(\theta, t_{-i}, R_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{t_{-i},R_{-i},m})) = a_{-i}\right\}\right], \tag{14}$$

$$\pi'_{i,m}\left[\theta, a_{-i}\right] = \mu'_{i,m}\left[\left\{\left(\theta, R_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{-i,R_{-i}})) = a_{-i}\right\}\right] \tag{15}$$

for every $\left(\theta, a_{-i}\right) \in \Theta \times A_{-i}$. Observe that $\left(\pi_{i,m}, \pi'_{i,m}\right) \in \Pi^{\mu_{i,m}}_i \times \Pi^{\mu'_{i,m}}_i$. Moreover,

$$\begin{aligned}\pi^*_{i,m}\left[\theta, a_{-i}\right] &= \kappa^*_{t_{i,m}}\left[\left\{\left(\theta, s_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(s_{-i})) = a_{-i}\right\}\right] \\ &= \frac{m}{m+1}\mu_{i,m}\left[\left\{\left(\theta, t_{-i}, R_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{t_{-i},R_{-i},m})) = a_{-i}\right\}\right] \\ &\quad + \frac{1}{m+1}\mu'_{i,m}\left[\left\{\left(\theta, R_{-i}\right) : \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(\tau_{-i,R_{-i}})) = a_{-i}\right\}\right] \\ &= \frac{m}{m+1}\pi_{i,m}\left[\theta, a_{-i}\right] + \frac{1}{m+1}\pi'_{i,m}\left[\theta, a_{-i}\right]. \end{aligned}$$

where the first equality follows from (13); the second follows from (11); the third follows from (14) and (15). Therefore, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, a_i is no worse than α_i against $\frac{m}{m+1}\pi_{i,m} + \frac{1}{m+1}\pi'_{i,m}$. By the usual duality argument, $a_i \in BR_i\left(\frac{m}{m+1}\hat{\pi}_{i,m} + \frac{1}{m+1}\hat{\pi}'_{i,m}\right)$ for some $(\hat{\pi}_{i,m}, \hat{\pi}'_{i,m}) \in \Pi_i^{\mu_{i,m}} \times \Pi_i^{\mu'_{i,m}}$. It then follows from (iii) in (7) that $a_i \in R_i$. ■

Lemma 4 *$\mathcal{S}_i^*(t_i) \subset \mathcal{S}_i(t_i)$ for finite type t_i.*

Proof We fix a finite model (T, κ). We assume without loss of generality that (T, κ) is embedded in the universal type space (T^*, κ^*). We prove the claim by showing that for each $i \in I$, $t_i \in T_i$, $R_i \in \mathcal{S}_i^*(t_i)$, and $\varepsilon \in (0, 1]$, there exists $(\mu_i, \mu'_i) \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right) \times \Delta\left(\Theta \times \mathcal{R}_{-i}^{\uparrow}\right)$ such that

(i) $\mathrm{marg}_{\Theta \times T_{-i}} \mu_i = \kappa_{t_i}$;

(ii) $\mu_i\left[\left\{(\theta, t_{-i}, R_{-i}) : R_{-i} \in \mathcal{S}_{-i}^*(t_{-i})\right\}\right] = 1$;

(iii) $R_i \supset \bigcup_{(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_i}} BR_i\left((1-\varepsilon)\pi_i + \varepsilon\pi'_i\right)$.

Consequently, by (7), we have $\mathcal{S}_i^*(t_i) \subset \mathcal{S}_i(t_i)$.

First, since $R_i \in \mathcal{S}_i^*(t_i)$, there exist $\{t_{i,m}\}_{m=0}^{\infty} \subset T_i^*$ such that $t_{i,m} \to t_i$ and $R_i \supset ICR_i(t_{i,m})$ for every m. For each m, we define $\mu_{i,m} \in \Delta\left(\Theta \times T_{-i}^* \times \mathcal{R}_{-i}^{\uparrow}\right)$ by

$$\mu_{i,m}\left[\{\theta\} \times E_{-i} \times \{R_{-i}\}\right] = \kappa_{t_{i,m}}^*\left[\{(\theta, s_{-i}) : s_{-i} \in E_{-i} \text{ and } ICR_{-i}(s_{-i}) = R_{-i}\}\right] \tag{16}$$

for every measurable $E_{-i} \subset T_{-i}^*$ and $(\theta, R_{-i}) \in \Theta \times \mathcal{R}_{-i}^{\uparrow}$. Since $\Delta\left(\Theta \times T_{-i}^* \times \mathcal{R}_{-i}^{\uparrow}\right)$ is a weak* compact metric space, $\{\mu_{i,m}\}_{m=0}^{\infty}$ admits a convergent subsequence $\{\mu_{i,m_k}\}_{k=0}^{\infty}$. We denote its limit by μ_i. Second, we show that μ_i satisfies (i) and (ii). By the definition of $\mu_{i,m}$, we know that $\mathrm{marg}_{\Theta \times T_{-i}^*} \mu_{i,m} = \kappa_{t_{i,m}}$. Since $\mu_{i,m_k} \to \mu_i$ as $k \to \infty$ and $t_{i,m} \to t_i$ as $m \to \infty$, it follows that $\mathrm{marg}_{\Theta \times T_{-i}^*} \mu_i = \kappa_{t_i}$, i.e., (i) holds. In particular, we have $\mu_i \in \Delta\left(\Theta \times T_{-i} \times \mathcal{R}_{-i}^{\uparrow}\right)$.

To prove (ii), for each $\ell \in \mathbb{N}$, let

$$F_\ell = \mathrm{cl}\left\{(\theta, s_{-i}, R_{-i}) : \exists s'_{-i} \in T^*_{-i} \text{ s.t. } d_{-i}(s'_{-i}, s_{-i}) \leq \frac{1}{\ell} \text{ and } ICR_{-i}\left(s'_{-i}\right) = R_{-i}\right\},$$

$$F_\infty = (\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}) \cap \bigcap_{\ell \in \mathbb{N}} F_\ell,$$

where d_{-i} is the metric on T^*_{-i}. Note that

$$F_\ell \supset \left\{(\theta, t_{-i}, R_{-i}) : ICR_{-i}\left(t_{-i}\right) = R_{-i}\right\}, \forall \ell, \tag{17}$$

$$F_\infty \subset \left\{(\theta, t_{-i}, R_{-i}) : t_{-i} \in T_{-i} \text{ and } R_{-i} \in \mathcal{S}^*_{-i}(t_{-i})\right\}. \tag{18}$$

Hence, (16) and (17) imply that $\mu_{i,m}[F_\ell] \geq \mu_{i,m}\left[ICR_{-i}\left(t_{-i}\right) = R_{-i}\right] = 1$, i.e., $\mu_{i,m}[F_\ell] = 1$ for all ℓ. Since F_ℓ is closed and $\mu_{i,m_k} \to \mu_i$ as $k \to \infty$, we have $\mu_i[F_\ell] = 1$ for all ℓ. As a result, $\mu_i[\bigcap_{\ell \in \mathbb{N}} F_\ell] = 1$. Combining this with $\mu_i\left[\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right] = 1$, we have $\mu_i\left[F_\infty\right] = 1$, which, together with (18), implies (ii).

Finally, we prove (iii). First, let

$$\mu'_{i,m} = \frac{1}{\varepsilon}\left(\mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_{i,m} - (1-\varepsilon)\, \mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_i\right). \tag{19}$$

Since $\mu_{i,m_k} \to \mu_i$, pick k sufficiently large so that $\mu'_{i,m_k}\left[\theta, R_{-i}\right] \geq 0$ for every $(\theta, R_{-i}) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, and hence $\mu'_{i,m_k} \in \Delta\left(\Theta \times \mathcal{R}^{\uparrow}_{-i}\right)$. Now fix any $a_i \in A_i$ such that $a_i \in BR_i\left(\left(1-\varepsilon\right)\pi_i + \varepsilon\pi'_i\right)$ for some $(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_{i,m_k}}$, and we show $a_i \in R_i$. Fix $\alpha_i \in \Delta(A_i \setminus \{a_i\})$. For each $(\theta, R_{-i}) \in \Theta \times \mathcal{R}^{\uparrow}_{-i}$, define $\psi^{\alpha_i}_{-i}(\theta, R_{-i})$ as in (12). Then since $a_i \in BR_i\left(\left(1-\varepsilon\right)\pi_i + \varepsilon\pi'_i\right)$ for some $(\pi_i, \pi'_i) \in \Pi_i^{\mu_i} \times \Pi_i^{\mu'_{i,m_k}}$, it follows that

$$\int_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))] d\left(\left(1-\varepsilon\right) \mathrm{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_i + \varepsilon\mu'_{i,m_k}\right) \geq 0. \tag{20}$$

Let $\nu_{i,m_k} \in \Delta\left(\Theta \times T^*_{-i} \times A_{-i}\right)$ be such that

$$\nu_{i,m_k}\left[\{\theta\} \times E_{-i} \times \{a_{-i}\}\right] = \kappa^*_{t_{i,m_k}}\left[\left\{(\theta, t_{-i}) : t_{-i} \in E_{-i} \text{ and } \psi^{\alpha_i}_i(\theta, ICR_{-i}(t_{-i})) = a_{-i}\right\}\right] \tag{21}$$

for every measurable $E_{-i} \subset T^*_{-i}$ and $(\theta, a_{-i}) \in \Theta \times A_{-i}$. Since $\psi_i^{\alpha_i}(\theta, ICR_{-i}(t_{-i})) \in ICR_{-i}(t_{-i})$, ν_{i,m_k} is a valid conjecture. We then have

$$
\begin{aligned}
&\int_{\Theta \times T^*_{-i} \times A_{-i}} [u_i(\theta, a_i, a_{-i}) - u_i(\theta, \alpha_i, a_{-i})] d\nu_{i,m_k} \\
&= \int_{\Theta \times T^*_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(t_{-i}))) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, ICR_{-i}(t_{-i})))] d\kappa^*_{t_{i,m_k}} \\
&= \int_{\Theta \times T^*_{-i} \times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))] d\mu_{i,m_k} \\
&= \int_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} [u_i(\theta, a_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i})) - u_i(\theta, \alpha_i, \psi^{\alpha_i}_{-i}(\theta, R_{-i}))] d\left((1-\varepsilon) \operatorname{marg}_{\Theta \times \mathcal{R}^{\uparrow}_{-i}} \mu_i + \varepsilon \mu'_{i,m_k}\right) \\
&\geq 0,
\end{aligned}
$$

where the three equalities follow from (21), (16), and (19), respectively, and the inequality follows from (20). Therefore, for each $\alpha_i \in \Delta(A_i \setminus \{a_i\})$, there exists a valid conjecture ν_{i,m_k} for t_{i,m_k} against which a_i is no worse than α_i. Then it follows from the usual duality argument that we can find a valid conjecture for t_{i,m_k}, independent of α_i, against which a_i is a best reply. By (1), we have $a_i \in ICR_i\left(t_{i,m_k}\right) \subset R_i$. ■

A.4 Proof of Proposition 4

First, suppose that $\left(\widetilde{\mathcal{S}}_i\right)_{i \in I}$ is an $\mathcal{R}^{\uparrow}$-perturbed curb collection. Then, the fact that $\widetilde{\mathcal{S}}_i(t_i) \subset \mathcal{S}^*_i(t_i)$ follows from the proof of Lemma 3 in Appendix A.3 by noting that $\left(\widetilde{\mathcal{S}}_i\right)_{i \in I}$ satisfies the same fixed-point property as $(\mathcal{S}_i)_{i \in I}$ in (7); moreover, the measurability of $\tau_{t_{-i}, R_{-i}, m}$ on T^*_{-i} is ensured by the measurability of μ_{t_j, R_i} on t_j for every j.

Second, the fact that $(\mathcal{S}^*_i|_{T_i})_{i \in I}$ is an $\mathcal{R}^{\uparrow}$-perturbed curb collection follows from the proof of Lemma 4 in Appendix A.3 by adding the following step to ensure the measurability of μ: For each $t_i \in T_i$ and each $R_i \in \mathcal{S}^*_i(t_i)$, let Δ_{t_i, R_i} be the set of weak* limits of all $\mu_{t_{i,m}} \in \Delta\left(\Theta \times T^*_{-i} \times \mathcal{A}_{-i}\right)$ such that $\{t_{i,m}\} \to t_i$ and $S^{\infty}_i(t_{i,m}) = R_i$ for all m, where $\mu_{t_{i,m}}$ is defined as $\mu_{i,m}$ in (16). By the compactness of $\Delta(\Theta \times T^*_{-i} \times \mathcal{A}_{-i})$, we have $\Delta_{t_i, R_i} \neq \varnothing$. Also Δ_{t_i, R_i} depends on (t_i, R_i) upper hemicontinuously. Thus, it follows from the Kuratowski–Ryll-Nardzewski selection theorem that we have a measurable function $\mu : T_i \times \mathcal{A}_i \to \Delta\left(\Theta \times T_{-i} \times \mathcal{R}^{\uparrow}_{-i}\right)$ such that $\mu_{t_i,R_i} \in \Delta_{t_i,R_i}$ whenever $R_i \in \mathcal{S}^*_i(t_i)$.

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