Controlling versus enabling — Online appendix

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Section 1 shows the sense in which Proposition 1 and 2 in Section 4 of the main paper hold in a much more general setting. Section 2 shows that the principal can achieve the best possible outcome with a linear contract. Section 3 establishes the result stated at the end of Section 5.2 in the main paper, namely that Proposition 4 continues to hold even if prices are endogenous and contractible, and there are production costs. Section 4 provides the detailed working for the analysis in Section 6.1 of the main paper with liquidity constraints. Finally, Section 5 establishes the results stated in Section 6.3 in the main paper, which focuses on the case in which prices are the transferable decision variables.

1 Generalization of our model

In this section we extend our model to allow for general functional forms and any number of non-transferable actions (i.e. more than one of each type).

Denote by $R(\mathbf{a},\mathbf{q},\mathbf{Q})$ the revenue generated jointly by the principal and the agent if the latter accepts the principal's contract. The actions contained in the vector $\mathbf{a}=\left(a^1,..,a^{M_a}\right)$ are transferable, i.e. each of them can be chosen *either* by the principal *or* by the agent, depending on how the principal chooses to allocate control rights. The actions contained in the vectors $\mathbf{q}=\left(q^1,..,q^{M_q}\right)$ and $\mathbf{Q}=\left(Q^1,..,Q^{M_Q}\right)$ are non-transferable. I.e. the agent always chooses $\mathbf{q}\in\mathbb{R}_+^{M_q}$ at cost $c(\mathbf{q})\equiv\sum_{i=1}^{M_q}c^i\left(q^i\right)$ and the principal always chooses $\mathbf{Q}\in\mathbb{R}_+^{M_Q}$ at cost $C(\mathbf{Q})\equiv\sum_{i=1}^{M_Q}C^i\left(Q^i\right)$. For the transferable actions, if the principal chooses $a^i\in\mathbb{R}_+$, it incurs cost $F^i\left(a^i\right)$. If the agent chooses $a^i\in\mathbb{R}_+$, it incurs cost $f^i\left(a^i\right)=\theta^iF^i\left(a^i\right)$. We assume there is at least one action of each type, i.e. $M_q\geq1$, $M_Q\geq1$ and $M_a\geq1$.

The principal chooses the set $D\subset\{1,..,M_a\}$ of transferable decisions over which it keeps control

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(leaving the agent to control decisions $i \in \{1,..,M_a\}\setminus D$) and offers a revenue-sharing contract $\Omega(R)$ to the agent, where $\Omega(.)$ can be any arbitrary function of the revenue R generated. The contract means that the agent obtains $\Omega(R)$, while the principal obtains $R-\Omega(R)$.

Throughout this section, we use the following three notational conventions. First, for variables and parameters that apply to both the agent and the principal, we use lowercase for the agent and uppercase for the principal (e.g. $\mathbf{q}$ and $\mathbf{Q}$). Second, vectors are written in bold. Third, subscripts next to functions always indicate derivatives: for example, $F^i_{a^i}$ indicates the derivative of F^i with respect to a^i and R_{a^i} indicates the partial derivative of R with respect to the transferable action a^i.

We make the following technical assumptions:

(a1) All functions are twice continuously differentiable in all arguments.

(a2) For all $i \in \{1,..,M_a\}$, $j \in \{1,..,M_q\}$ and $k \in \{1,..,M_Q\}$, the revenue function $R(\mathbf{a},\mathbf{q},\mathbf{Q})$ is increasing in a^i, q^j and Q^k, the cost functions F^i, c^j and C^k are increasing and convex, and

$$F^i(0)=F^i_{a^i}(0)=c^j(0)=c^j_{q^j}(0)=C^k(0)=C^k_{Q^k}(0)=0.$$

(a3) For all $t \in [0,1]$ and $D \subset \{1,...,M_a\}$, $tR(\mathbf{a},\mathbf{q},\mathbf{Q})-\sum_{i\in D}F^i\left(a^i\right)-\sum_{i\in\{1,...,M_a\}\setminus D}\theta^i F^i\left(a^i\right)-c(\mathbf{q})-C(\mathbf{Q})$ is concave in $(\mathbf{a},\mathbf{q},\mathbf{Q})$ and admits a unique finite maximizer in any subset of the $M_a+M_q+M_Q$ variables $(\mathbf{a},\mathbf{q},\mathbf{Q})$ for any values of the remaining variables.

(a4) For all $\boldsymbol{\tau} \in [0,1]^{M_a+M_q+M_Q}$, the system of equations

$$\begin{cases} \tau^i R_{a^i}(\mathbf{a},\mathbf{q},\mathbf{Q}) = F^i_{a^i}\left(a^i\right) & \text{for } i \in D \\ \tau^i R_{a^i}(\mathbf{a},\mathbf{q},\mathbf{Q}) = \theta^i F^i_{a^i}\left(a^i\right) & \text{for } i \in \{1,..,M_a\}\setminus D \\ \tau^{M_a+j} R_{q^j}(\mathbf{a},\mathbf{q},\mathbf{Q}) = c^j_{q^j}\left(q^j\right) & \text{for } j \in \{1,..,M_q\} \\ \tau^{M_a+M_q+k} R_{Q^k}(\mathbf{a},\mathbf{q},\mathbf{Q}) = C^k_{Q^k}\left(Q^k\right) & \text{for } k \in \{1,..,M_Q\} \end{cases} \tag{1}$$

admits a unique solution $(\mathbf{a}(\boldsymbol{\tau}),\mathbf{q}(\boldsymbol{\tau}),\mathbf{Q}(\boldsymbol{\tau}))$.

These assumptions are standard and ensure that the optimization problems considered below are well-behaved. Assumptions (a3) and (a4) ensure that there is always a unique finite solution to the optimization problems we consider; in particular, they obviate the need for more general stability conditions for uniqueness, that would be quite complex in this setting. Furthermore, the principal always finds it optimal to induce the agent to participate.

In the next section, we establish that in this set-up, we can restrict attention to linear contracts without loss of generality. This result implies that we can restrict attention to contracts offered by the principal that take the form

$$\Omega(R) = (1-t)R - T,$$

where T can be interpreted as the fixed fee collected by the principal and $t \in [0,1]$ as the share of revenue kept by the principal. Then given an allocation of decision rights $D \subset \{1,..,M_a\}$ chosen by the principal, its profits can be written as[1]

$$\Pi^*(D) = \max_{t,\mathbf{a},\mathbf{q},\mathbf{Q}} \left\{ R(\mathbf{a},\mathbf{q},\mathbf{Q}) - \sum_{i\in D} F^i\left(a^i\right) - \sum_{i\in\{1,...,M_a\}\setminus D} \theta^i F^i\left(a^i\right) - c(\mathbf{q}) - C(\mathbf{Q}) \right\} \tag{2}$$

s.t.

$$\begin{cases} tR_{a^i}(\mathbf{a},\mathbf{q},\mathbf{Q}) = F^i_{a^i}\left(a^i\right) \text{ for } i \in D \\ (1-t)R_{a^i}(\mathbf{a},\mathbf{q},\mathbf{Q}) = \theta^i F^i_{a^i}\left(a^i\right) \text{ for } i \in \{1,..,M_a\}\setminus D \\ (1-t)R_{q^j}(\mathbf{a},\mathbf{q},\mathbf{Q}) = c^j_{q^j}\left(q^j\right) \text{ for } j \in \{1,..,M_q\} \\ tR_{Q^k}(\mathbf{a},\mathbf{q},\mathbf{Q}) = C^k_{Q^k}\left(Q^k\right) \text{ for } k \in \{1,..,M_Q\}. \end{cases} \tag{3}$$

For future reference, denote by $t^{\mathcal{P}*}$ and $t^{\mathcal{A}*}$ the respective optimal variable fees charged by the principal in $\mathcal{P}$-mode and $\mathcal{A}$-mode, i.e. the respective solutions in t that emerge from (2)-(3) when $D = \{1,..,M_a\}$ and $D = \emptyset$. Also let

$$\Pi^{\mathcal{P}*} \equiv \Pi^*(\{1,..,M_a\}) \text{ and } \Pi^{\mathcal{A}*} \equiv \Pi^*(\emptyset).$$

The analysis in the rest of this section will rely on the following additional assumption:

(a5) For $D \subset \{1,..,M_a\}$ and $\boldsymbol{\tau} \in [0,1)^{M_a+M_q+M_Q}$, if

$$\Pi(D,\boldsymbol{\tau}) \equiv R(\mathbf{a}(\boldsymbol{\tau}),\mathbf{q}(\boldsymbol{\tau}),\mathbf{Q}(\boldsymbol{\tau})) - \sum_{i\in D} F^i\left(a^i(\boldsymbol{\tau})\right) - \sum_{i\in\{1,...,M_a\}\setminus D} \theta^i F^i\left(a^i(\boldsymbol{\tau})\right) - c(\mathbf{q}(\boldsymbol{\tau})) - C(\mathbf{Q}(\boldsymbol{\tau})),$$

where $(\mathbf{a}(\boldsymbol{\tau}),\mathbf{q}(\boldsymbol{\tau}),\mathbf{Q}(\boldsymbol{\tau}))$ is the unique solution to the system of equations (1), then $\Pi(D,\boldsymbol{\tau})$ is increasing in each τ^i for $i \in \{1,..,M_a+M_q+M_Q\}$.

In words, this assumption requires that reducing the distortion in any second stage decision problem for any action (by increasing τ^i) increases the principal's overall net profit. It can be shown that a sufficient condition for (a5) to hold is that $R(\mathbf{a},\mathbf{q},\mathbf{Q})$ is weakly supermodular in all of its arguments.[2]

[1] At the optimum, the fixed fee T of the linear contract is always set such that the participation constraint of the agent is binding, i.e.

$$(1-t)R(\mathbf{a},\mathbf{q},\mathbf{Q}) - \sum_{i\in\{1,..,M_a\}\setminus D} f^i\left(a^i\right) - c(\mathbf{q}) - T = 0.$$

[2] Details are available from the authors.

However, weak supermodularity is not necessary: (a5) can still hold even when the various non-contractible actions are strategic substitutes in the revenue function. What is required in this case is that the interaction effects are not too negative, so that they do not overwhelm the direct positive effect on net profits of increasing the incentive to invest in any given non-contractible action by raising the corresponding τ^i. Seen in this light, (a5) is a rather mild assumption.

We can then show that the main results from Propositions 1 and 2 in the main paper continue to hold in this more general setting.

Proposition 1. *Suppose assumptions (a1)-(a5) hold and that when* $\left(\theta^1,...,\theta^{M_a}\right) = (1,...,1)$*, the optimal revenue share* t^* *is different from* $1/2$*. Then the principal's optimal contract satisfies the following properties:*

1. *If* θ^i *is sufficiently close to* 1 *for all* $i \in \{1,..,M_a\}$*, it is optimal to give control over all* M_a *transferable actions to the same party. I.e. there exist* $\left(\varepsilon^1,..,\varepsilon^{M_a}\right) \in (0,1)^{M_a}$ *such that one of the pure modes (*$\mathcal{P}$*-mode or* $\mathcal{A}$*-mode) is optimal whenever* $\left|\theta^i - 1\right| \leq \varepsilon^i$ *for all* $i \in \{1,..,M_a\}$*.*

2. *Suppose* $\left(\theta^1,...,\theta^{M_a}\right) = (1,...,1)$*. If the principal optimally sets* $t^* < 1/2$*, then the* $\mathcal{A}$*-mode is strictly optimal (i.e.* $\Pi^{\mathcal{A}*} > \Pi^{\mathcal{P}*}$*); if* $t^* > 1/2$*, then the* $\mathcal{P}$*-mode is strictly optimal (i.e.* $\Pi^{\mathcal{P}*} > \Pi^{\mathcal{A}*}$*).*

Proof. Let $\tau(D,t)$ be the vector of $M_a + M_q + M_Q$ coordinates defined as follows:

$$\tau^i(D,t) = \begin{cases} t & \text{if} \quad i \in D \cup \{M_a + M_q + 1,..,M_a + M_q + M_Q\} \\ 1-t & \text{if} \quad i \in (\{1,..,M_a\} \backslash D) \cup \{M_a + 1,..,M_a + M_q\}. \end{cases}$$

Then the profit obtained by the principal is equal to $\Pi(D, \tau(D,t))$, where $\Pi(D,\tau)$ is defined in assumption (a5) above.

Suppose first $\left(\theta^1,...,\theta^{M_a}\right) = (1,...,1)$. Then it is easily seen that $\Pi(D,\tau)$ only depends on τ, but not on D—we therefore denote it by $\Pi(\tau)$. Denote by t^* the optimal variable fee and by $(D^*, \{1,..,M_a\} \backslash D^*)$ the optimal allocation of control rights over the transferable actions. Suppose $D^* \neq \emptyset$ and $D^* \neq \{1,..,M_a\}$. If $t^* < 1 - t^*$ (i.e. $t^* < 1/2$), then the principal could increase profits by giving up control over all actions a_j for $j \in D^*$ to the agent and keeping t^* unchanged. To see this, note that the change in profits is $\Pi(\boldsymbol{\tau}(\emptyset, t^*)) - \Pi(\boldsymbol{\tau}(D^*, t^*))$. If $t^* > 0$, then (a5) implies this difference is positive, because $0 < t^* < 1 - t^* < 1$ and $D^* \neq \emptyset$ imply $\boldsymbol{\tau}(D^*, t^*) \in [0,1)^{M_a+M_q+M_Q}$, $\boldsymbol{\tau}(\emptyset, t^*) \in [0,1)^{M_a+M_q+M_Q}$ and $\boldsymbol{\tau}(\emptyset, t^*) > \boldsymbol{\tau}(D^*, t^*)$. If $t^* = 0$, then the change in profits can be written

$$\Pi\left(\boldsymbol{\tau}\left(\emptyset,0\right)\right)-\Pi\left(\boldsymbol{\tau}\left(D^{*},0\right)\right) = \max_{\mathbf{a},\mathbf{q}}\left\{R\left(\mathbf{a},\mathbf{q},\mathbf{Q}=\mathbf{0}\right)-\sum_{i\in\{1,\ldots,M_a\}}F^{i}\left(a^{i}\right)-c\left(\mathbf{q}\right)\right\} - \left(\begin{array}{c}\max_{\mathbf{a},\mathbf{q}}\left\{R\left(\mathbf{a},\mathbf{q},\mathbf{Q}=\mathbf{0}\right)-\sum_{i\in\{1,\ldots,M_a\}}F^{i}\left(a^{i}\right)-c\left(\mathbf{q}\right)\right\}\\ \text{s.t. } a^{i}=0 \text{ if } i\in D^{*}\end{array}\right).$$

In this case, $D^* \neq \emptyset$ and (a1)-(a4) imply $\Pi\left(\boldsymbol{\tau}\left(\emptyset,0\right)\right)-\Pi\left(\boldsymbol{\tau}\left(D^{*},0\right)\right) > 0$. In particular, this implies the $\mathcal{A}$-mode is optimal and dominates the $\mathcal{P}$-mode.

By a symmetric argument, if $t^* > 1-t^*$ (i.e. $t^* > 1/2$), then the principal could increase profits by taking control over all actions $j \in \{1,..,M_a\}\backslash D^*$ and keeping t^* unchanged. In particular, this implies the $\mathcal{P}$-mode is optimal and dominates the $\mathcal{A}$-mode.

Thus, we have proven that, when $\left(\theta^1,...,\theta^{M_a}\right)=(1,...,1)$, either one of the pure modes, i.e. $D^*=\emptyset$ (i.e. $\mathcal{A}$-mode) or $D^*=\{1,..,M_a\}$ (i.e. $\mathcal{P}$-mode), strictly dominates all other $2^{M_a}-2$ allocations of control rights, provided $t^* \neq 1/2$. By continuity of all profit functions in $\left(\theta^1,...,\theta^{M_a}\right)$, this remains true when $\left(\theta^1,...,\theta^{M_a}\right)$ is in a neighborhood of $(1,...,1)$.

□

2 Optimality of linear contracts

In this section we show that, even with the more general model setup of Section 1, the principal can achieve the best outcome with linear contracts. I.e., the principal can restrict attention, without loss of generality, to contracts of the form

$$\Omega\left(R\right)=\left(1-t\right)R-T,$$

where T can be interpreted as the fixed fee collected by the principal and $t\in[0,1]$ as the share of revenue kept by the principal.

Given an allocation of control rights $D \subset \{1,..,M_a\}$, the principal's optimal contract $\Omega^*(.)$ solves

$$\begin{aligned}
\Pi^*(D) \quad = \quad & \max_{\Omega(.),\mathbf{Q},\mathbf{a},\mathbf{q}} \left\{ R(\mathbf{a},\mathbf{q},\mathbf{Q}) - \Omega(R(\mathbf{a},\mathbf{q},\mathbf{Q})) - \sum_{i\in D} F^i\left(a^i\right) - \sum_{i\in\{1,\dots,M_a\}\setminus D} \theta^i F^i\left(a^i\right) - C(\mathbf{Q}) \right\} \\
& \text{s.t.} \\
& a^i = \arg\max_a \left\{ \begin{array}{c} R\left(a^1,..,a^{i-1},a,a^{i+1},..,a^{M_a},\mathbf{q},\mathbf{Q}\right) \\ -\Omega\left(R\left(a^1,..,a^{i-1},a,a^{i+1},..,a^{M_a},\mathbf{q},\mathbf{Q}\right)\right) - F^i(a) \end{array} \right\} \text{ for } i \in D \\
& a^i = \arg\max_a \left\{\Omega\left(R\left(a^1,..,a^{i-1},a,a^{i+1},..,a^{M_a},\mathbf{q},\mathbf{Q}\right)\right) - \theta^i F^i(a)\right\} \text{ for } i \in \{1,..,M_a\}\setminus D \\
& \mathbf{q} = \arg\max_{\mathbf{q}'} \left\{\Omega\left(R\left(\mathbf{a},\mathbf{q}',\mathbf{Q}\right)\right) - c\left(\mathbf{q}'\right)\right\} \\
& \mathbf{Q} = \arg\max_{\mathbf{Q}'} \left\{R\left(\mathbf{a},\mathbf{q},\mathbf{Q}'\right) - \Omega\left(R\left(\mathbf{a},\mathbf{q},\mathbf{Q}'\right)\right) - C\left(\mathbf{Q}'\right)\right\} \\
& 0 \le \Omega(R(\mathbf{a},\mathbf{q},\mathbf{Q})) - \textstyle\sum_{i\in\{1,\dots,M_a\}\setminus D} \theta^i F^i\left(a^i\right) - c(\mathbf{q}).
\end{aligned}$$

Let $(\mathbf{a}^*,\mathbf{q}^*,\mathbf{Q}^*)$ denote the outcome of this optimization problem and define $R^* \equiv R(\mathbf{a}^*,\mathbf{q}^*,\mathbf{Q}^*)$.

We first prove that $\Omega^*(.)$ must be continuous and differentiable at R^*.

Suppose the optimal allocation of decision rights is $D^* \subset \{1,..,M_a\}$, the proposed optimal contract Ω^* offered by the principal is discontinuous at R^* and $\lim_{R\to R^{*-}} \Omega^*(R) > \lim_{R\to R^{*+}} \Omega^*(R)$. Then

$$\mathbf{Q}^* = \arg\max_{\mathbf{Q}} \{R(\mathbf{a}^*,\mathbf{q}^*,\mathbf{Q}) - \Omega^*(R(\mathbf{a}^*,\mathbf{q}^*,\mathbf{Q})) - C(\mathbf{Q})\}$$

implies $\Omega^*(R^*) = \lim_{R\to R^{*+}} \Omega^*(R)$, because otherwise $\Omega^*(R^*) > \lim_{R\to R^{*+}} \Omega^*(R)$, so the principal could profitably deviate to, say, $Q^{*1}+\varepsilon$, with ε sufficiently small. But then we must have $a^i = 0$ for all $i \in \{1,..,M_a\}\setminus D^*$ and $\mathbf{q}^* = 0$, since otherwise the agent could profitably deviate to $a^i - \varepsilon$ for some $i \in \{1,..,M_a\}\setminus D^*$ or $q^{*j} - \varepsilon$ for some $j \in \{1,..,M_q\}$, with ε sufficiently small. If $a^i = 0$ for all $i \in \{1,..,M_a\}\setminus D^*$ and $\mathbf{q}^* = 0$, then it must be that $\Omega^*(R^*) = 0$ and $D^* = \{1,..,M_a\}$ (i.e. pure $\mathcal{P}$-mode), and therefore

$$(\mathbf{a}^*,\mathbf{Q}^*) = \arg\max_{\mathbf{a},\mathbf{Q}} \left\{ R(\mathbf{a},\mathbf{0},\mathbf{Q}) - \sum_{i=1}^{M_a} F^i\left(a^i\right) - C(\mathbf{Q}) \right\}. \tag{4}$$

This also means the principal's profits are

$$\Pi^* = \max_{\mathbf{a},\mathbf{Q}} \left\{ R(\mathbf{a},\mathbf{0},\mathbf{Q}) - \sum_{i=1}^{M_a} F^i\left(a^i\right) - C(\mathbf{Q}) \right\}.$$

In this case the principal could keep $D^* = \{1,..,M_a\}$ but switch to the following linear contract

$$\Omega_\varepsilon(R) = \varepsilon R + c(\mathbf{q}(\varepsilon)) - \varepsilon R(\mathbf{a}(\varepsilon),\mathbf{q}(\varepsilon),\mathbf{Q}(\varepsilon)),$$

where $\varepsilon > 0$ is sufficiently small and $(\mathbf{a}(\varepsilon), \mathbf{q}(\varepsilon), \mathbf{Q}(\varepsilon))$ is a solution to

$$\begin{cases} \mathbf{a}(\varepsilon) = \arg\max_{\mathbf{a}} \left\{(1-\varepsilon) R(\mathbf{a}, \mathbf{q}(\varepsilon), \mathbf{Q}(\varepsilon)) - \sum_{i=1}^{M_a} F^i\left(a^i\right)\right\} \\ \mathbf{q}(\varepsilon) = \arg\max_{\mathbf{q}} \{\varepsilon R(\mathbf{a}(\varepsilon), \mathbf{q}, \mathbf{Q}(\varepsilon)) - c(\mathbf{q})\} \\ \mathbf{Q}(\varepsilon) = \arg\max_{\mathbf{Q}} \{(1-\varepsilon) R(\mathbf{a}(\varepsilon), \mathbf{q}(\varepsilon), \mathbf{Q}) - C(\mathbf{Q})\}. \end{cases}$$

Denote the principal's profit that results from $D^* = \{1, .., M_a\}$ and contract Ω_ε by

$$\Pi^{\mathcal{P}}(\varepsilon) \equiv R(\mathbf{a}(\varepsilon), \mathbf{q}(\varepsilon), \mathbf{Q}(\varepsilon)) - \sum_{i=1}^{M_a} F^i\left(a^i(\varepsilon)\right) - c(\mathbf{q}(\varepsilon)) - C(\mathbf{Q}(\varepsilon)).$$

Clearly, $(\mathbf{a}(0), \mathbf{q}(0), \mathbf{Q}(0)) = (\mathbf{a}^*, \mathbf{0}, \mathbf{Q}^*)$ and $\Pi^{\mathcal{P}}(0) = \Pi^*$. We can then use (4), the definition of $\mathbf{q}(\varepsilon)$ and assumption (a2) to obtain

$$\begin{aligned} \Pi^{\mathcal{P}}_\varepsilon(0) &= \sum_{i=1}^{M_a} \left(R_{a^i}(\mathbf{a}^*, \mathbf{0}, \mathbf{Q}^*) - F^i_{a^i}\left(a^{*i}\right)\right) a^i_\varepsilon(0) + \sum_{j=1}^{M_q} R_{q^j}(\mathbf{a}^*, \mathbf{0}, \mathbf{Q}^*)\, q^j_\varepsilon(0) \\ &\quad + \sum_{k=1}^{M_Q} \left(R_{Q^k}(\mathbf{a}^*, \mathbf{0}, \mathbf{Q}^*) - C^k_{Q^k}\left(Q^{*k}\right)\right) Q^k_\varepsilon(0) \\ &= \sum_{j=1}^{M_q} \frac{R_{q^j}(\mathbf{a}^*, \mathbf{0}, \mathbf{Q}^*)^2}{c^j_{q^j q^j}(0)} > 0. \end{aligned}$$

Thus, if $\lim_{R \to R^{*-}} \Omega^*(R) > \lim_{R \to R^{*+}} \Omega^*(R)$, then the principal can keep $D^* = \{1, .., M_a\}$ but profitably deviate to $\Omega_\varepsilon(R)$ for ε small enough, which contradicts the optimality of $\Omega^*(R)$.

The other possibility is $\lim_{R \to R^{*+}} \Omega^*(R) > \lim_{R \to R^{*-}} \Omega^*(R)$. Then

$$\mathbf{q}^* = \arg\max_{\mathbf{q}} \{\Omega^*(R(\mathbf{a}^*, \mathbf{q}, \mathbf{Q}^*)) - c(\mathbf{q})\}$$

implies $\Omega^*(R^*) = \lim_{R \to R^{*+}} \Omega^*(R)$, because otherwise $\Omega^*(R^*) < \lim_{R \to R^{*+}} \Omega^*(R)$, so the agent could profitably deviate to, say, $q^{*1} + \varepsilon$, with ε sufficiently small. But then we must have $a^i = 0$ for all $i \in D^*$ and $\mathbf{Q}^* = 0$, otherwise the principal could profitably deviate to $a^i - \varepsilon$ for some $i \in D^*$ or $Q^{*k} - \varepsilon$ for some $k \in \{1, .., M_Q\}$, with ε sufficiently small. This implies $D^* = \emptyset$ and thus, the principal's profits are at most

$$\Pi^* \leq R(\mathbf{a}^*, \mathbf{q}^*, \mathbf{0}) - \sum_{i=1}^{M_a} \theta^i F^i\left(a^{i*}(\varepsilon)\right) - c(\mathbf{q}^*) \leq \max_{\mathbf{a}, \mathbf{q}} \left\{R(\mathbf{a}, \mathbf{q}, \mathbf{0}) - \sum_{i=1}^{M_a} \theta^i F^i\left(a^i(\varepsilon)\right) - c(\mathbf{q})\right\}.$$

This cannot be optimal. Indeed, the principal could keep $D^* = \emptyset$ and switch to the linear contract

$$\widetilde{\Omega}_\varepsilon(R) = (1-\varepsilon)R + \sum_{i=1}^{M_a} \theta^i F^i\left(\widetilde{a}^i(\varepsilon)\right) + c\left(\widetilde{\mathbf{q}}(\varepsilon)\right) - (1-\varepsilon)R\left(\widetilde{\mathbf{a}}(\varepsilon), \widetilde{\mathbf{q}}(\varepsilon), \widetilde{\mathbf{Q}}(\varepsilon)\right),$$

where $\varepsilon > 0$ is sufficiently small and $\left(\widetilde{\mathbf{a}}(\varepsilon), \widetilde{\mathbf{q}}(\varepsilon), \widetilde{\mathbf{Q}}(\varepsilon)\right)$ is a solution to

$$\begin{cases} \widetilde{\mathbf{a}}(\varepsilon) = \arg\max_a \left\{(1-\varepsilon)R\left(\mathbf{a}, \widetilde{\mathbf{q}}(\varepsilon), \widetilde{\mathbf{Q}}(\varepsilon)\right) - \sum_{i=1}^{M_a} \theta^i F^i\left(a^i(\varepsilon)\right)\right\} \\ \widetilde{\mathbf{q}}(\varepsilon) = \arg\max_q \left\{(1-\varepsilon)R\left(\widetilde{\mathbf{a}}(\varepsilon), \mathbf{q}, \widetilde{\mathbf{Q}}(\varepsilon)\right) - c(\mathbf{q})\right\} \\ \widetilde{\mathbf{Q}}(\varepsilon) = \arg\max_Q \left\{\varepsilon R\left(\widetilde{\mathbf{a}}(\varepsilon), \widetilde{\mathbf{q}}(\varepsilon), \mathbf{Q}\right) - C(\mathbf{Q})\right\}. \end{cases}$$

Denote the principal's profit that results from offering contract $\widetilde{\Omega}_\varepsilon$ by

$$\widetilde{\Pi}(\varepsilon) \equiv R\left(\widetilde{\mathbf{a}}(\varepsilon), \widetilde{\mathbf{q}}(\varepsilon), \widetilde{\mathbf{Q}}(\varepsilon)\right) - \sum_{i=1}^{M_a} \theta^i F^i\left(\widetilde{a}^i(\varepsilon)\right) - c\left(\widetilde{\mathbf{q}}(\varepsilon)\right) - C\left(\widetilde{\mathbf{Q}}(\varepsilon)\right).$$

Clearly,

$$\begin{aligned} \left(\widetilde{\mathbf{a}}(0), \widetilde{\mathbf{q}}(0), \widetilde{\mathbf{Q}}(0)\right) &= \left(\widetilde{\mathbf{a}}^*, \widetilde{\mathbf{q}}^*, \mathbf{0}\right) \equiv \arg\max_{\mathbf{a},\mathbf{q}} \left\{R(\mathbf{a}, \mathbf{q}, \mathbf{0}) - \sum_{i=1}^{M_a} \theta^i F^i\left(a^i\right) - c(\mathbf{q})\right\} \\ \widetilde{\Pi}(0) &= \max_{\mathbf{a},\mathbf{q}} \left\{R(\mathbf{a}, \mathbf{q}, \mathbf{0}) - \sum_{i=1}^{M_a} \theta^i F^i\left(a^i\right) - c(\mathbf{q})\right\} \geq \Pi^*. \end{aligned}$$

Using the last inequality, the definitions of $\widetilde{\mathbf{a}}(\varepsilon)$ and $\widetilde{\mathbf{Q}}(\varepsilon)$ and assumption (a2), we obtain

$$\begin{aligned} \widetilde{\Pi}_\varepsilon(0) &= \sum_{i=1}^{M_a} \left(R_{a^i}\left(\widetilde{\mathbf{a}}^*, \widetilde{\mathbf{q}}^*, \mathbf{0}\right) - \theta^i F^i_{a^i}\left(\widetilde{a}^{*i}\right)\right)\widetilde{a}^i_\varepsilon(0) + \sum_{j=1}^{M_q} \left(R_{q^j}\left(\widetilde{\mathbf{a}}^*, \widetilde{\mathbf{q}}^*, \mathbf{0}\right) - c^j_{q^j}\left(\widetilde{q}^{*j}\right)\right)\widetilde{q}^j_\varepsilon(0) \\ &\quad + \sum_{k=1}^{M_Q} R_{Q^k}\left(\widetilde{\mathbf{a}}^*, \widetilde{\mathbf{q}}^*, \mathbf{0}\right)\widetilde{Q}^k_\varepsilon(0) \\ &= \sum_{k=1}^{M_Q} \frac{R_{Q^k}\left(\widetilde{\mathbf{a}}^*, \widetilde{\mathbf{q}}^*, \mathbf{0}\right)^2}{C^k_{Q^kQ^k}(0)} > 0. \end{aligned}$$

Thus, the principal can profitably deviate to $\widetilde{\Omega}_\varepsilon(R)$ for ε small enough, which contradicts the optimality of $\Omega^*(R)$.

We have thus proven that $\lim_{R\to R^{*+}} \Omega^*(R) = \lim_{R\to R^{*-}} \Omega^*(R)$, so Ω^* is continuous at R^*.

Suppose now that Ω^* is non-differentiable at R^* and $\lim_{R\to R^{*+}} \Omega^*_R(R) > \lim_{R\to R^{*-}} \Omega^*_R(R)$. This implies $a^i = 0$ for all $i \in \{1,..,M_a\} \backslash D^*$ and $\mathbf{q}^* = \mathbf{0}$, otherwise there exist $i \in \{1,..,M_a\} \backslash D^*$

such that $a^{i*} > 0$ and $j \in \{1,..,M_q\}$ such that $q^{*j} > 0$. In this case, setting a^i slightly below a^{*i} and q^j slightly below q^{*j} would violate $a^{*i} = \arg\max_a \left\{\Omega^*\left(R\left(\mathbf{a}^*_{-i}(a), \mathbf{q}^*, \mathbf{Q}^*\right)\right) - \theta^i F^i(a)\right\}$ and $q^{*j} = \arg\max_q \left\{\Omega^*\left(R\left(\mathbf{a}^*, \mathbf{q}^*_{-j}(q), \mathbf{Q}^*\right)\right) - c^j(q)\right\}$, where

$$\begin{aligned} \mathbf{a}^*_{-i}(a) &\equiv \left(a^{*1},..,a^{*(i-1)},a,a^{*(i+1)},..,a^{*M_a}\right) \\ \mathbf{q}^*_{-j}(q) &\equiv \left(q^{*1},..,q^{*(j-1)},q,q^{*(j+1)},..,q^{*M_q}\right). \end{aligned}$$

Indeed, we would then have

$$\begin{aligned} 0 &\geq \lim_{a^i \to a^{*i+}} \left\{\Omega^*_R\left(R\left(\mathbf{a}^*_{-i}(a), \mathbf{q}^*, \mathbf{Q}^*\right)\right) R_{a^i}\left(\mathbf{a}^*_{-i}(a), \mathbf{q}^*, \mathbf{Q}^*\right) - \theta^i F^i_{a^i}\left(a^i\right)\right\} \\ &> \lim_{a^i \to a^{*i-}} \left\{\Omega^*_R\left(R\left(\mathbf{a}^*_{-i}(a), \mathbf{q}^*, \mathbf{Q}^*\right)\right) R_{a^i}\left(\mathbf{a}^*_{-i}(a), \mathbf{q}^*, \mathbf{Q}^*\right) - \theta^i F^i_{a^i}\left(a^i\right)\right\} \end{aligned}$$

and

$$\begin{aligned} 0 &\geq \lim_{q^j \to q^{*j+}} \left\{\Omega^*_R\left(R\left(\mathbf{a}^*, \mathbf{q}^*_{-j}\left(q^j\right), \mathbf{Q}^*\right)\right) R_{q^j}\left(\mathbf{a}^*, \mathbf{q}^*_{-j}\left(q^j\right), \mathbf{Q}^*\right) - c^j_{q^j}\left(q^j\right)\right\} \\ &> \lim_{q^j \to q^{*j-}} \left\{\Omega^*_R\left(R\left(\mathbf{a}^*, \mathbf{q}^*_{-j}\left(q^j\right), \mathbf{Q}^*\right)\right) R_{q^j}\left(\mathbf{a}^*, \mathbf{q}^*_{-j}\left(q^j\right), \mathbf{Q}^*\right) - c^j_{q^j}\left(q^j\right)\right\}. \end{aligned}$$

But $a^i = 0$ for all $i \in \{1,..,M_a\} \setminus D^*$ and $\mathbf{q}^* = \mathbf{0}$ implies that we must have $\Omega^*(R^*) = 0$ (recall $c(\mathbf{0}) = F^i(0) = 0$) and $D^* = \{1,..,M_a\}$, so

$$(\mathbf{a}^*, \mathbf{Q}^*) = \arg\max_{\mathbf{a},\mathbf{Q}} \left\{R(\mathbf{a}, \mathbf{0}, \mathbf{Q}) - \sum_{i=1}^{M_a} F^i\left(a^i\right) - C(\mathbf{Q})\right\}.$$

We can then apply the same reasoning as above to conclude that the principal could profitably deviate to the linear contract $\Omega_\varepsilon(R)$ for ε small enough.

Suppose instead $\lim_{R \to R^{*+}} \Omega^*_R(R) < \lim_{R \to R^{*-}} \Omega^*_R(R)$. By a very similar reasoning to the one above, this implies $a^i = 0$ for all $i \in D^*$ and $\mathbf{Q}^* = \mathbf{0}$. But $\mathbf{Q}^* = \mathbf{0}$ implies that

$$\Pi^* \leq \max_{\mathbf{a},\mathbf{q}} \{R(\mathbf{a}, \mathbf{q}, \mathbf{0}) - f(\mathbf{a}) - c(\mathbf{q})\}.$$

We can then apply the same reasoning as above to conclude that the principal could profitably deviate to $D = \emptyset$ and the linear contract $\widetilde{\Omega}_\varepsilon(R)$ for ε small enough.

We conclude that $\Omega^*(.)$ must be continuous and differentiable at R^*. This result implies that

$(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*)$ solve

$$\begin{cases} \left(1 - \Omega_R^*(R^*)\right) R_{a^i}(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*) = F_{a^i}^i\left(a^{*i}\right) \text{ for } i \in D \\ \Omega_R^*(R^*) R_{a^i}(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*) = \theta^i F_{a^i}^i\left(a^{*i}\right) \text{ for } i \in \{1, .., M_a\} \setminus D \\ \Omega_R^*(R^*) R_{q^j}(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*) = c_{q^j}^j\left(q^{*j}\right) \text{ for } j \in \{1, .., M_q\} \\ \left(1 - \Omega_R^*(R^*)\right) R_{Q^k}(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*) = C_{Q^k}\left(Q^{k*}\right) \text{ for } k \in \{1, .., M_Q\}. \end{cases}$$

Let $t^* \equiv 1 - \Omega_R^*(R^*)$ and $T^* \equiv (1 - t^*) R^* - \Omega^*(R^*)$. Clearly, the linear contract $\widehat{\Omega}(R) = (1 - t^*) R - T^*$ can generate the same stage-2 symmetric Nash equilibrium $(\mathbf{a}^*, \mathbf{q}^*, \mathbf{Q}^*)$ as the initial contract $\Omega^*(R)$. Furthermore, both $\Omega^*(R)$ and $\widehat{\Omega}(R)$ cause the agents' participation constraint to bind and therefore result in the same profits for the principal.

3 Endogenous price and production costs

We now extend the model with spillovers by allowing the principal to also set a price in the contracting stage, along with the fees (t, T), and by also adding a production cost. We will establish the result stated at the end of Section 5.2 in the main paper, i.e. that Proposition 4 continues to hold in this case.

The revenue generated by agent i is now

$$R(p, a_i, q_i, Q) = (p - c)\left(D_0 + \beta a_i + x\left(\overline{a}_{-i} - a_i\right) + \phi q_i + \Phi Q - p\right),$$

where $c \geq 0$ is a constant marginal production cost, p is the price chosen by the principal and D_0 is some baseline level of demand.

First, we show that whether the production cost is incurred by the principal or the agent does not affect profits in either mode. In $\mathcal{P}$-mode, if the principal incurs the production cost, then the maximization problem is[3]

$$\widetilde{\Pi}^{\mathcal{P}*} = \max_{p,t,a,q,Q} \left\{ N \left((p-c)(D_0 + \beta a + \phi q + \Phi Q - p) - \frac{1}{2}a^2 - \frac{1}{2}q^2 \right) - \frac{1}{2}Q^2 \right\}$$
$$\text{s.t.} \quad \begin{cases} (tp - c)\beta = a \\ (1-t)\phi p = q \\ (tp-c) N\Phi = Q. \end{cases}$$

If instead the agent incurs the production cost, then the maximization problem is

$$\widetilde{\Pi}^{\mathcal{P}*} = \max_{p,\widetilde{t},a,q,Q} \left\{ N \left((p-c)(D_0 + \beta a + \phi q + \Phi Q - p) - \frac{1}{2}a^2 - \frac{1}{2}q^2 \right) - \frac{1}{2}Q^2 \right\}$$
$$\text{s.t.} \quad \begin{cases} \widetilde{t}p\beta = a \\ \left(\left(1-\widetilde{t}\right)p - c\right)\phi = q \\ \widetilde{t}pN\Phi = Q. \end{cases}$$

By making the change of variables $\widetilde{t} \equiv t - \frac{c}{p}$, the second maximization problem becomes the same as the first.

Similarly, in $\mathcal{A}$-mode, if the principal incurs the production cost, then the maximization problem is

$$\widetilde{\Pi}^{\mathcal{A}*} = \max_{p,t,a,q,Q} \left\{ N \left((p-c)(D_0 + \beta a + \phi q + \Phi Q - p) - \frac{1}{2}a^2 - \frac{1}{2}q^2 \right) - \frac{1}{2}Q^2 \right\}$$
$$\text{s.t.} \quad \begin{cases} (1-t)p(\beta - x) = a \\ (1-t)p\Phi = q \\ (tp-c) N\Phi = Q. \end{cases}$$

[3]The analysis that follows would be identical if we allowed for spillovers across the choices of prices. These spillovers would have no impact on the resulting tradeoff because they are internalized in both modes by the principal when it sets prices in the contracting stage.

If instead the agent incurs the production cost, then the maximization problem is

$$
\begin{aligned}
\widetilde{\Pi}^{\mathcal{A}*} \quad &= \quad \max_{p,\widetilde{t},a,q,Q} \left\{ N \left((p-c)(D_0 + \beta a + \phi q + \Phi Q - p) - \frac{1}{2}a^2 - \frac{1}{2}q^2 \right) - \frac{1}{2}Q^2 \right\} \\
&\quad \text{s.t.} \\
&\quad \left\{ \begin{array}{c} \left(\left(1-\widetilde{t}\right)p - c\right)(\beta - x) = a \\ \left(\left(1-\widetilde{t}\right)p - c\right)\phi = q \\ \widetilde{t}pN\Phi = Q. \end{array} \right.
\end{aligned}
$$

Again, by making the change of variables $\widetilde{t} = t - \frac{c}{p}$, the second maximization problem becomes the same as the first. Thus, in our setting it is irrelevant which party actually incurs the production cost.

Solving the program above in $\mathcal{P}$-mode, we obtain

$$
\widetilde{\Pi}^{\mathcal{P}*} = N \max_{p,t} \left\{ (p-c)(D_0 - p) + \frac{\beta^2 + N\Phi^2}{2}(tp-c)((2-t)p-c) + \frac{\phi^2}{2}p(1-t)(p(1+t)-2c) \right\}.
$$

Holding p fixed and optimizing over t, we obtain

$$
t^{\mathcal{P}*}(p) = \frac{\left(\beta^2 + N\Phi^2\right)p + \phi^2 c}{\left(\beta^2 + N\Phi^2 + \phi^2\right)p}.
$$

Substituting this back into $\widetilde{\Pi}^{\mathcal{P}*}$, the program becomes

$$
\widetilde{\Pi}^{\mathcal{P}*} = \max_{p} \left\{ N(p-c)(D_0 - p) + (p-c)^2 \Pi^{\mathcal{P}*} \right\},
$$

where $\Pi^{\mathcal{P}*}$ is given by

$$
\Pi^{\mathcal{P}*} = \frac{N}{2}\left(\beta^2 + N\Phi^2 + \frac{\phi^4}{\beta^2 + \phi^2 + N\Phi^2} \right).
$$

Similarly, solving the program above in $\mathcal{A}$-mode, we have

$$
\widetilde{\Pi}^{\mathcal{A}*} = N \max_{p,t} \left\{ \begin{array}{c} (p-c)(D_0 - p) + \frac{N\Phi^2}{2}(tp-c)((2-t)p-c) \\ + \frac{\phi^2}{2}p(1-t)((1+t)p-2c) + \frac{1}{2}(1-t)p(\beta - x)((\beta + x + (\beta - x)t)p - 2\beta c) \end{array} \right\}.
$$

Holding p fixed and optimizing over t, we obtain

$$
t^{\mathcal{A}*}(p) = \frac{N\Phi^2 p + \phi^2 c + (\beta - x)(\beta c - xp)}{\left(N\Phi^2 + \phi^2 + (\beta - x)^2\right)p}
$$

Substituting this back into $\widetilde{\Pi}^{\mathcal{A}*}$, the program becomes (after straightforward calculations)

$$\widetilde{\Pi}^{\mathcal{A}*} = \max_{p} \left\{ (p-c)(D_0 - p) + (p-c)^2 \Pi^{\mathcal{A}*} \right\},$$

where $\Pi^{\mathcal{A}*}$ is given by

$$\Pi^{\mathcal{A}*} = \frac{N}{2}\left(\beta^2 - x^2 + \phi^2 + \frac{\left(N\Phi^2 - x(\beta - x)\right)^2}{(\beta - x)^2 + \phi^2 + N\Phi^2} \right).$$

Comparing the last expressions of $\widetilde{\Pi}^{\mathcal{A}*}$ and $\widetilde{\Pi}^{\mathcal{P}*}$, we can conclude that

$$\widetilde{\Pi}^{\mathcal{A}*} > \widetilde{\Pi}^{\mathcal{P}*} \Longleftrightarrow \Pi^{\mathcal{A}*} > \Pi^{\mathcal{P}*} \Longleftrightarrow \left| \frac{\phi^2 x}{\beta} + \beta^2 + N\Phi^2 \right| \leq \sqrt{\beta^2 \left(\beta^2 + \phi^2 + N\Phi^2\right) + \phi^4},$$

so the introduction of p and c does not affect the trade-off determined in Proposition 4 in the main paper.

4 Worker benefits and employees vs. contractors

Fix $\beta = 1$, $\Phi = 0.5$, and $B = 0.5$. We will consider the two numerical examples in turn.

4.1 Example with $\phi = 1.5$ and $W_0 = 1.5$

First consider the case with the correct classification but with a liquidity constraint. In $\mathcal{P}$-mode, we have

$$\begin{aligned} a(t) &= \beta t \\ q(t) &= \phi(1-t) \\ Q(t) &= \Phi t \end{aligned}$$

and the principal chooses t to maximize

$$t\left(\beta a(t) + \phi q(t) + \Phi Q(t)\right) - \frac{1}{2} a(t)^2 - \frac{1}{2} Q(t)^2 - B + T$$

s.t.

$$\begin{aligned} B + (1-t)\left(\beta a(t) + \phi q(t) + \Phi Q(t)\right) - \frac{1}{2} q(t)^2 - T &\geq W_0 \\ T &\leq 0. \end{aligned}$$

Here W_0 is the agent's payoff in the outside option.

Suppose the constraint that $T \leq 0$ is not binding. After substituting in T from the other constraint, this is equivalent to choosing t to maximize

$$\beta a(t) + \phi q(t) + \Phi Q(t) - \frac{1}{2} a(t)^2 - \frac{1}{2} q(t)^2 - \frac{1}{2} Q(t)^2 - W_0$$

which implies

$$t^P = \frac{\beta^2 + \Phi^2}{\beta^2 + \Phi^2 + \phi^2} < 1.$$

The principal's profit in $\mathcal{P}$-mode is

$$\Pi^P = \frac{1}{2}\left(\beta^2 + \Phi^2 + \frac{\phi^4}{\beta^2 + \Phi^2 + \phi^2}\right) - W_0. \tag{5}$$

In our example with $\phi = 1.5$ and $W_0 = 1.5$, the principal's profit in P-mode must therefore be negative.

Now consider $\mathcal{A}$-mode. In $\mathcal{A}$-mode we get that

$$\begin{aligned} a(t) &= \beta(1-t) \\ q(t) &= \phi(1-t) \\ Q(t) &= \Phi t \end{aligned}$$

and the principal chooses t to maximize

$$t\left(\beta a(t) + \phi q(t) + \Phi Q(t)\right) - \frac{1}{2} Q(t)^2 + T$$

s.t.

$$\begin{aligned} (1-t)\left(\beta a(t) + \phi q(t) + \Phi Q(t)\right) - \frac{1}{2} a(t)^2 - \frac{1}{2} q(t)^2 - T &\geq W_0 \qquad (6) \\ T &\leq 0. \end{aligned}$$

Suppose the constraint that $T \leq 0$ is not binding. After substituting in T from (6), this is equivalent to choosing t to maximize

$$\beta a(t) + \phi q(t) + \Phi Q(t) - \frac{1}{2}a(t)^2 - \frac{1}{2}q(t)^2 - \frac{1}{2}Q(t)^2 - W_0,$$

which implies

$$t^A = \frac{\Phi^2}{\beta^2 + \Phi^2 + \phi^2}. \tag{7}$$

This is always less than t^P and less than $\frac{1}{2}$ given $\phi > \Phi$. The agent gets more of the variable revenue. The principal's resulting profit is

$$\Pi^A = \frac{1}{2}\left(\beta^2 + \phi^2 + \frac{\Phi^4}{\beta^2 + \Phi^2 + \phi^2}\right) - W_0.$$

This profit is higher than in (5) given that $\phi > \Phi$. In our example with $\phi = 1.5$ and $W_0 = 1.5$, we get $\Pi^A = 0.134$. Moreover, solving for T given t^A from (6) implies $T = -0.082$, which means the principal makes a fixed transfer to the agent to get it to participate.

Now suppose the mode is incorrectly classified, so the principal has to provide the benefit B even if it allows the agent to choose a. This has no affect on the $\mathcal{P}$-mode profits, which are therefore still negative. In $\mathcal{A}$-mode, the analysis remains unchanged, except that W_0 is replaced by $W_0 - B$ and we also have to subtract B from the principal's resulting profit. That is, the principal chooses t to maximize

$$t(\beta a(t) + \phi q(t) + \Phi Q(t)) - \frac{1}{2}Q(t)^2 + T - B$$

s.t.

$$\begin{aligned} B + (1-t)(\beta a(t) + \phi q(t) + \Phi Q(t)) - \frac{1}{2}a(t)^2 - \frac{1}{2}q(t)^2 - T &\geq W_0 \\ T &\leq 0. \end{aligned} \tag{8}$$

Note that at t^A, the highest T that satisfies (8) is now positive ($T = 0.418$) reflecting that the agent now gets the additional benefit B and so doesn't need a positive transfer to be willing to participate. However, given the liquidity constraint, a positive T is not possible. Therefore, the optimal solution will involve $T = 0$.

With $T = 0$, the principal's problem in $\mathcal{A}$-mode is to choose t to maximize

$$t(\beta a(t) + \phi q(t) + \Phi Q(t)) - \frac{1}{2}Q(t)^2 - B$$

s.t.

$$B + (1-t)(\beta a(t) + \phi q(t) + \Phi Q(t)) - \frac{1}{2}a(t)^2 - \frac{1}{2}q(t)^2 \geq W_0.$$

That is, t is set to maximize

$$t(1-t)\left(\beta^2+\phi^2\right) + \frac{1}{2}\Phi^2 t^2 - B \tag{9}$$

s.t.

$$\frac{1}{2}(1-t)\left(\beta^2+\phi^2+\left(2\Phi^2-\beta^2-\phi^2\right)t\right) \geq W_0 - B. \tag{10}$$

The solution is either $t_0^A = \frac{\beta^2+\phi^2}{2(\beta^2+\phi^2)-\Phi^2}$, which is between $\frac{1}{2}$ and 1 given $\phi > \Phi$, provided this ensures (10) is satisfied, or if not, the value of t solving

$$\frac{1}{2}(1-t)\left(\beta^2+\phi^2+\left(2\Phi^2-\beta^2-\phi^2\right)t\right) = W_0 - B,$$

which we denote $t^A(W_0 - B)$. For our parameter values, (10) is not satisfied at t_0^A, so the principal does best setting $t = t^A(W_o - B) = 0.233$ and $T = 0$, such that the agent just wishes to participate. The principal's resulting profit is $\Pi^A = 0.088$.

4.2 Example with $\phi = 3$ and $W_0 = 4$

Consider first the case that the mode is correctly classified. We can make use of the above general results.

From (5) we have that $\Pi^P = 0.576$. Obviously, this is an upper bound on profits in $\mathcal{P}$-mode since we ignored the liquidity constraint in obtaining (5). In $\mathcal{A}$-mode, ignoring the liquidity constraint we have from (7) that $t^A = 0.024$. The agent's participation constraint (6) implies $T = 0.765$, which violates the liquidity constraint. Therefore $T = 0$ in $\mathcal{A}$-mode. The principal's problem is given by (9)-(10) with $B = 0$. For our parameter values, (10) with $B = 0$ is not satisfied at t_0^A, so the principal does best setting $t = t^A(W_o) = 0.108$ and $T = 0$, such that the agent just wishes to participate. The principal's profit is given by (9) with $B = 0$, which implies $\Pi^A = 0.967$, so the principal prefers the $\mathcal{A}$-mode.

Now suppose the mode is incorrectly classified, so the principal has to provide the benefit B even if it allows the agent to choose a. This reduces the revenue it needs to leave the agent to keep the agent participating. Since it can't set a positive fixed fee, it instead increases t. For our parameter values, (10) is not satisfied at t_0^A, so the principal does best setting $t = t^A(W_o) = 0.168$ and $T = 0$, such that the agent just wishes to participate. The principal's resulting profit is given by (9), which implies $\Pi^A = 0.898$.

5 Price as transferable decision and spillovers

Recall the revenue function

$$R_i\left(p_i, \overline{p}_{-i}, q_i, Q\right) = p_i\left(d + \beta p_i + x\left(\overline{p}_{-i} - p_i\right) + \phi q_i + \Phi Q\right)$$

and the assumptions made on parameters in the main paper:

$$\begin{aligned} \beta &< 0,\, \phi > 0,\, \Phi > 0 \\ -2\beta + \min\{0, 2x\} &> \max\left\{N\Phi^2, \phi^2\right\}. \end{aligned} \tag{11}$$

The fixed costs of agents' investment and principal's investment are quadratic:

$$c(q) = \frac{1}{2}q^2,\;\; C(Q) = \frac{1}{2}Q^2.$$

We want to prove the following proposition (Proposition 6 in the main paper):

Proposition. *The principal prefers the $\mathcal{A}$-mode if and only if*

$$-\frac{4k(k+\beta)}{k+2\beta} < x < 0. \tag{12}$$

Note that $0 < k < -\beta$ so $-\frac{4k(k+\beta)}{k+2\beta} < 0$. Furthermore, the proposition identifies a meaningful tradeoff since any positive x and any x satisfying (12) also satisfy (11) provided β is sufficiently negative.

In $\mathcal{P}$-mode, the payoff to agent i from working for the principal is

$$(1-t)\,R_i\left(p_i, \overline{p}_{-i}, q_i, Q\right) - \frac{1}{2}q_i^2 - T = (1-t)\,p_i\left(d + \beta p_i + x\left(\overline{p}_{-i} - p_i\right) + \phi q_i + \Phi Q\right) - \frac{1}{2}q_i^2 - T,$$

which the agent optimizes over q_i in the second stage (the fixed fee T is then taken as fixed).

The principal's payoff in the second stage is

$$\sum_{i=1}^{N}\left(tp_i\left(d + \beta p_i + x\left(\overline{p}_{-i} - p_i\right) + \phi q_i + \Phi Q\right)\right) - \frac{1}{2}Q^2,$$

which the principal optimizes over p_i and Q.

Evaluating the corresponding first-order conditions at the symmetric equilibrium, we have

$$\begin{cases} -2\beta p^{\mathcal{P}} = d + \phi q^{\mathcal{P}} + \Phi Q^{\mathcal{P}} \\ q^{\mathcal{P}} = (1-t)\,\phi p^{\mathcal{P}} \\ Q^{\mathcal{P}} = tN\Phi p^{\mathcal{P}}. \end{cases}$$

Solving, we obtain

$$\begin{aligned} p^{\mathcal{P}}(t) &= \tfrac{d}{-2\beta-(1-t)\phi^2-tN\Phi^2} \\ q^{\mathcal{P}}(t) &= \tfrac{d(1-t)}{-2\beta-(1-t)\phi^2-tN\Phi^2} \\ Q^{\mathcal{P}}(t) &= \tfrac{dtN}{-2\beta-(1-t)\phi^2-tN\Phi^2}. \end{aligned}$$

Note that assumptions (11) ensure $p^{\mathcal{P}}(t) > 0$, $q^{\mathcal{P}}(t) > 0$ and $Q^{\mathcal{P}}(t) > 0$.

The fixed fee T is just a transfer that renders each agent indifferent between working for the principal and their outside option, so the principal's profit is

$$\Pi^{\mathcal{P}}(t) = Np^{\mathcal{P}}(t)\left(d + \beta p^{\mathcal{P}}(t) + \phi q^{\mathcal{P}}(t) + \Phi Q^{\mathcal{P}}(t)\right) - N\frac{1}{2}q^{\mathcal{P}}(t)^2 - \frac{1}{2}Q^{\mathcal{P}}(t)^2.$$

Plugging in the expressions of $p^{\mathcal{P}}(t)$, $q^{\mathcal{P}}(t)$ and $Q^{\mathcal{P}}(t)$ above, we obtain:

$$\Pi^{\mathcal{P}}(t) = \max_t \left\{ \frac{Nd^2\left(-2\beta - (1-t)^2\phi^2 - t^2N\Phi^2\right)}{2\left(-2\beta - (1-t)\,\phi^2 - tN\Phi^2\right)^2} \right\}. \tag{13}$$

In $\mathcal{A}$-mode, agent i joining the principal chooses (p_i, q_i) to maximize his second stage payoff

$$(1-t)\,p_i\left(d + \beta p_i + x\left(\overline{p}_{-i} - p_i\right) + \phi q_i + \Phi Q\right) - \frac{1}{2}q_i^2,$$

while the principal chooses Q to maximize its second stage revenues

$$\sum_{i=1}^{N} tp_i\left(d + \beta p_i + x\left(\overline{p}_{-i} - p_i\right) + \phi q_i + \Phi Q\right) - \frac{1}{2}Q^2.$$

Evaluating the corresponding first-order conditions at the symmetric equilibrium, we have

$$\begin{cases} (-2\beta + x)\,p^{\mathcal{A}} = d + \phi q^{\mathcal{A}} + \Phi Q^{\mathcal{A}} \\ q^{\mathcal{A}} = (1-t)\,\phi p^{\mathcal{A}} \\ Q^{\mathcal{A}} = tN\Phi p^{\mathcal{A}}. \end{cases}$$

Solving, we obtain

$$\begin{aligned} p^{\mathcal{A}}(t) &= \tfrac{d}{-2\beta+x-(1-t)\phi^2-tN\Phi^2} \\ q^{\mathcal{A}}(t) &= \tfrac{d(1-t)}{-2\beta+x-(1-t)\phi^2-tN\Phi^2} \\ Q^{\mathcal{A}}(t) &= \tfrac{dtN}{-2\beta+x-(1-t)\phi^2-tN\Phi^2}. \end{aligned}$$

Assumptions (11) ensure $p^{\mathcal{A}}(t) > 0$, $q^{\mathcal{A}}(t) > 0$ and $Q^{\mathcal{A}}(t) > 0$.

The fixed fee T renders each agent indifferent between joining the principal and his outside option, so the principal's profit in $\mathcal{A}$-mode is

$$\Pi^{\mathcal{A}}(t) = Np^{\mathcal{A}}(t)\left(d + \beta p^{\mathcal{A}}(t) + \phi q^{\mathcal{A}}(t) + \Phi Q^{\mathcal{A}}(t)\right) - N\frac{1}{2}q^{\mathcal{A}}(t)^2 - \frac{1}{2}Q^{\mathcal{A}}(t)^2.$$

Plugging in the expressions of $p^{\mathcal{A}}(t)$, $q^{\mathcal{A}}(t)$ and $Q^{\mathcal{A}}(t)$ above, we obtain:

$$\Pi^{\mathcal{A}}(t) = \max_t \left\{ \frac{Nd^2\left(2(-\beta+x) - (1-t)^2\phi^2 - t^2N\Phi^2\right)}{2\left(-2\beta + x - (1-t)\phi^2 - tN\Phi^2\right)^2} \right\}. \tag{14}$$

Comparing expressions (13) and (14), $\Pi^{\mathcal{P}}(t)$ is obtained from $\Pi^{\mathcal{A}}(t)$ simply by setting $x = 0$. Therefore, we will focus on maximizing $\Pi^{\mathcal{A}}(t)$, from which we can easily derive the maximization of $\Pi^{\mathcal{P}}(t)$.

The first-order derivative of $\Pi^{\mathcal{A}}(t)$ in t is proportional to (with a strictly positive multiplying factor)

$$N\Phi(-2\beta + 2x) - \phi^2 x - N\Phi^2\phi^2 - t\left(\left(N\Phi^2 + \phi^2\right)(-2\beta + x) - 2N\Phi^2\phi^2\right).$$

Since $\left(N\Phi^2 + \phi^2\right)(-2\beta + x) - 2N\Phi^2\phi^2 > 0$ under assumptions (11), we obtain that the optimal variable fee under the $\mathcal{A}$-mode is

$$t^{\mathcal{A}*} = \begin{cases} 0 & \text{if } N\Phi^2(-2\beta+2x) - \phi^2 x - N\Phi^2\phi^2 \le 0 \\ \frac{N\Phi^2(-2\beta+2x)-\phi^2x-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta+x)-2N\Phi^2\phi^2} & \text{if } 0 \le N\Phi^2(-2\beta+2x) - \phi^2 x - N\Phi^2\phi^2 \\ & \le \left(N\Phi^2+\phi^2\right)(-2\beta+x) - 2N\Phi^2\phi^2 \\ 1 & \text{if } N\Phi^2(-2\beta+2x) - \phi^2 x - N\Phi^2\phi^2 \\ & \ge \left(N\Phi^2+\phi^2\right)(-2\beta+x) - 2N\Phi^2\phi^2. \end{cases}$$

Rewriting the conditions:

$$t^{\mathcal{A}*} = \begin{cases} 0 & \text{if } x\left(\phi^2 - 2N\Phi^2\right) \geq N\Phi^2\left(-2\beta - \phi^2\right) \\ \frac{N\Phi^2(-2\beta+2x)-\phi^2 x - N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta+x)-2N\Phi^2\phi^2} & \text{if } x\left(\phi^2 - 2N\Phi^2\right) \leq N\Phi^2\left(-2\beta - \phi^2\right) \\ & \text{and } x\left(N\Phi^2 - 2\phi^2\right) \leq \phi^2\left(-2\beta - N\Phi^2\right) \\ 1 & \text{if } x\left(N\Phi^2 - 2\phi^2\right) \geq \phi^2\left(-2\beta - N\Phi^2\right). \end{cases}$$

Suppose x is such that $0 < t^{\mathcal{A}*} < 1$. Then the first-order condition of $\Pi^{\mathcal{A}}(t)$ in t evaluated at $t^{\mathcal{A}*}$ implies:

$$\left(\left(1 - t^{\mathcal{A}*}\right)\phi^2 - t^{\mathcal{A}*}N\Phi^2\right)\left(\begin{array}{c} -2\beta + x - \left(1 - t^{\mathcal{A}*}\right)\phi^2 \\ -t^{\mathcal{A}*}N\Phi^2 \end{array}\right) = \left(\phi^2 - N\Phi^2\right)\left(\begin{array}{c} 2\left(-\beta + x\right) - \left(1 - t^{\mathcal{A}*}\right)^2\phi^2 \\ -\left(t^{\mathcal{A}*}\right)^2 N\Phi^2 \end{array}\right),$$

from which we can deduce:

$$\begin{aligned} \Pi^{\mathcal{A}} &= \frac{Nd^2\left(2\left(-\beta + x\right) - \left(1 - t^{\mathcal{A}*}\right)^2\phi^2 - \left(t^{\mathcal{A}*}\right)^2 N\Phi^2\right)}{2\left(-2\beta + x - \left(1 - t^{\mathcal{A}*}\right)\phi^2 - t^{\mathcal{A}*}N\Phi^2\right)^2} \\ &= \frac{Nd^2\left(\left(1 - t^{\mathcal{A}*}\right)\phi^2 - t^{\mathcal{A}*}N\Phi^2\right)}{2\left(\phi^2 - N\Phi^2\right)\left(-2\beta + x - \left(1 - t^{\mathcal{A}*}\right)\phi^2 - t^{\mathcal{A}*}N\Phi^2\right)} \\ &= \frac{Nd^2}{2\left(\phi^2 - N\Phi^2\right)}\frac{\phi^2 - t^{\mathcal{A}*}\left(N\Phi^2 + \phi^2\right)}{-2\beta + x - \phi^2 + t^{\mathcal{A}*}\left(\phi^2 - N\Phi^2\right)}. \end{aligned}$$

Plugging $t^{\mathcal{A}*} = \frac{N\Phi^2(-2\beta+2x)-\phi^2 x - N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta+x)-2N\Phi^2\phi^2}$ into the last expression, we obtain

$$\Pi^{\mathcal{A}*} = \frac{Nd^2}{2}\frac{\left(-2\beta + 2x\right)\left(N\Phi^2 + \phi^2\right) - N\Phi^2\phi^2}{\left(N\Phi^2 + \phi^2\right)\left(-2\beta - N\Phi^2 + x\right)\left(-2\beta - \phi^2 + x\right) - x\left(N\Phi^2 - \phi^2\right)^2}.$$

From here, we can set $x = 0$ to obtain

$$\begin{aligned} t^{\mathcal{P}*} &= \frac{\left(-2\beta - \phi^2\right)N\Phi^2}{-2\beta\left(N\Phi^2 + \phi^2\right) - 2N\Phi^2\phi^2} \in (0, 1) \\ \Pi^{\mathcal{P}*} &= \frac{Nd^2}{2}\frac{-2\beta\left(N\Phi^2 + \phi^2\right) - N\Phi^2\phi^2}{\left(N\Phi^2 + \phi^2\right)\left(-2\beta - N\Phi^2\right)\left(-2\beta - \phi^2\right)}. \end{aligned}$$

The complete characterization of profits in $\mathcal{A}$-mode is:

$$\Pi^{\mathcal{A}*}=\begin{cases}\frac{Nd^2}{2}\frac{2(-\beta+x)-\phi^2}{(-2\beta+x-\phi^2)^2} & \text{if } x\left(\phi^2-2N\Phi^2\right)\geq N\Phi^2\left(-2\beta-\phi^2\right)\\ \frac{Nd^2}{2}\frac{(-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta-N\Phi^2+x)(-2\beta-\phi^2+x)-x(N\Phi^2-\phi^2)^2} & \begin{array}{c}\text{if } x\left(\phi^2-2N\Phi^2\right)\leq N\Phi^2\left(-2\beta-\phi^2\right)\\ \text{and } x\left(N\Phi^2-2\phi^2\right)\leq\phi^2\left(-2\beta-N\Phi^2\right)\end{array}\\ \frac{Nd^2}{2}\frac{2(-\beta+x)-N\Phi^2}{(-2\beta+x-N\Phi^2)^2} & \text{if } x\left(N\Phi^2-2\phi^2\right)\geq\phi^2\left(-2\beta-N\Phi^2\right).\end{cases}$$

Suppose $x\left(\phi^2-2N\Phi^2\right)\leq N\Phi^2\left(-2\beta-\phi^2\right)$ and $x\left(N\Phi^2-2\phi^2\right)\leq\phi^2\left(-2\beta-N\Phi^2\right)$, so that $0<t^{\mathcal{A}*}<1$. We have $\Pi^{\mathcal{A}}>\Pi^{\mathcal{P}}$ if and only if

$$\frac{(-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta-N\Phi^2+x)(-2\beta-\phi^2+x)-x\left(N\Phi^2-\phi^2\right)^2}>\frac{-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta-N\Phi^2)(-2\beta-\phi^2)},$$

which is equivalent to

$$\begin{aligned}&\left((-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(N\Phi^2+\phi^2\right)\left(-2\beta-N\Phi^2\right)\left(-2\beta-\phi^2\right)\\ >\;&\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(\left(N\Phi^2+\phi^2\right)\left(-2\beta-N\Phi^2+x\right)\left(-2\beta-\phi^2+x\right)-x\left(N\Phi^2-\phi^2\right)^2\right).\end{aligned}$$

Recall the two sides are equal for $x=0$, therefore we can eliminate all terms that are not factored by x or x^2, so the inequality reduces to

$$\begin{aligned}&2x\left(N\Phi^2+\phi^2\right)^2\left(-2\beta-N\Phi^2\right)\left(-2\beta-\phi^2\right)\\ >\;&\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(-x\left(N\Phi^2-\phi^2\right)^2+x\left(N\Phi^2+\phi^2\right)\left(-4\beta-\left(N\Phi^2+\phi^2\right)\right)+x^2\left(N\Phi^2+\phi^2\right)\right).\end{aligned}$$

Rearranging, this can be rewritten

$$\begin{aligned}0\;>\;&-x\begin{pmatrix}\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(2\left(N^2\Phi^4+\phi^4\right)+4\beta\left(N\Phi^2+\phi^2\right)\right)\\ +2\left(N\Phi^2+\phi^2\right)^2\left(-2\beta-N\Phi^2\right)\left(-2\beta-\phi^2\right)\end{pmatrix}+\\ &+x^2\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(N\Phi^2+\phi^2\right).\end{aligned}$$

Simplifying, this leads to

$$0>-2xN\Phi^2\phi^2\left(2\beta\left(N\Phi^2+\phi^2\right)+2N\Phi^2\phi^2\right)+x^2\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(N\Phi^2+\phi^2\right),$$

from which we conclude

$$\Pi^{\mathcal{A}*} > \Pi^{\mathcal{P}*} \Longleftrightarrow x\left(\frac{2N\Phi^2\phi^2\left(-2\beta\left(N\Phi^2+\phi^2\right)-2N\Phi^2\phi^2\right)}{\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(N\Phi^2+\phi^2\right)}+x\right)<0.$$

Both the numerator and the denominator of the large fraction are positive under assumptions (11).

We conclude that when $0 < t^{\mathcal{A}*} < 1$:

$$\begin{aligned}\Pi^{\mathcal{A}*} \quad > \quad \Pi^{\mathcal{P}*} &\Longleftrightarrow -\frac{2N\Phi^2\phi^2\left(-2\beta\left(N\Phi^2+\phi^2\right)-2N\Phi^2\phi^2\right)}{\left(-2\beta\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2\right)\left(N\Phi^2+\phi^2\right)}<x<0\\ &\Longleftrightarrow \quad -\frac{4\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\left(\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\right)}{2\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}}<x<0\end{aligned}$$

It remains to consider the cases $x\left(\phi^2-2N\Phi^2\right) \geq N\Phi^2\left(-2\beta-\phi^2\right)$ (in which $t^{\mathcal{A}*}=0$) and $x\left(N\Phi^2-2\phi^2\right) \geq \phi^2\left(-2\beta-N\Phi^2\right)$ (in which $t^{\mathcal{A}*}=1$). It is easier to consider the following three cases in turn.

Case I: $\phi^2 > 2N\Phi^2$.

In this case, it is easily verified that assumptions (11) imply $x\left(N\Phi^2-2\phi^2\right) \leq \phi^2\left(-2\beta-N\Phi^2\right)$. Therefore we have:

$$\Pi^{\mathcal{A}*}=\begin{cases}\frac{Nd^2}{2}\frac{2(-\beta+x)-\phi^2}{\left(-2\beta+x-\phi^2\right)^2} & \text{if } x \geq \frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2}\\ \frac{Nd^2}{2}\frac{(-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta-N\Phi^2+x)(-2\beta-\phi^2+x)-x(N\Phi^2-\phi^2)^2} & \text{if } \frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2} \geq x \geq -\frac{-2\beta-\max\left\{\phi^2,N\Phi^2\right\}}{2}\end{cases}.$$

The expression $\frac{2(-\beta+x)-\phi^2}{\left(-2\beta+x-\phi^2\right)^2}$ is increasing in x for $x \leq 0$ and decreasing in x for $x \geq 0$, therefore the maximum value attained by $\Pi^{\mathcal{A}}$ when $x \geq \frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2}$ is precisely when $x = \frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2}$. That value is:

$$\begin{aligned}\Pi^{\mathcal{A}*}\left(x=\frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2}\right) &= \frac{Nd^2}{2}\frac{\phi^2\left(\phi^2-2N\Phi^2\right)}{\left(-2\beta-\phi^2\right)\left(\phi^2-N\Phi^2\right)^2}\\ &< \frac{Nd^2}{2}\frac{(-2\beta)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{\left(N\Phi^2+\phi^2\right)\left(-2\beta-N\Phi^2\right)\left(-2\beta-\phi^2\right)}=\Pi^{\mathcal{P}*},\end{aligned}$$

where the inequality is straightforward to verify under assumptions (11). Thus, $\Pi^{\mathcal{P}*}$ dominates $\Pi^{\mathcal{A}*}$ for all $x \geq \frac{N\Phi^2\left(-2\beta-\phi^2\right)}{\phi^2-2N\Phi^2}$. Combining with the result above, we conclude that $\Pi^{\mathcal{P}*}$ dominates $\Pi^{\mathcal{A}*}$ for all $x \geq 0$ and $x \leq -\frac{4\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\left(\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\right)}{2\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}}$, whereas $\Pi^{\mathcal{A}*}$ dominates $\Pi^{\mathcal{P}*}$ for all permissible x such that

$$-\frac{4\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\left(\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\right)}{2\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}} \leq x \leq 0.$$

Case II: $N\Phi^2 > 2\phi^2$.

In this case, it is easily verified that assumptions (11) imply $x\left(\phi^2 - 2N\Phi^2\right) \leq N\Phi^2\left(-2\beta - \phi^2\right)$. Therefore we have:

$$\Pi^{\mathcal{A}*} = \begin{cases} \frac{Nd^2}{2}\frac{2(-\beta+x)-N\Phi^2}{(-2\beta+x-N\Phi^2)^2} & \text{if } x \geq \frac{\phi^2\left(-2\beta-N\Phi^2\right)}{N\Phi^2-2\phi^2} \\ \frac{Nd^2}{2}\frac{(-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{(N\Phi^2+\phi^2)(-2\beta-N\Phi^2+x)(-2\beta-\phi^2+x)-x(N\Phi^2-\phi^2)^2} & \text{if } \frac{\phi^2\left(-2\beta-N\Phi^2\right)}{N\Phi^2-2\phi^2} \geq x \geq -\frac{-2\beta-\max\left\{\phi^2,N\Phi^2\right\}}{2} \end{cases}.$$

The analysis is exactly the same as in Case I above (by symmetry in ϕ^2 and $N\Phi^2$), therefore the conclusion is exactly the same for this case as well.

Case III: $\phi^2 \leq 2N\Phi^2$ and $N\Phi^2 \leq 2\phi^2$.

In this case, it is easily verified that assumptions (11) imply $x\left(N\Phi^2 - 2\phi^2\right) \leq \phi^2\left(-2\beta - N\Phi^2\right)$ and $x\left(\phi^2 - 2N\Phi^2\right) \leq N\Phi^2\left(-2\beta - \phi^2\right)$ for all permissible x. Therefore we have:

$$\Pi^{\mathcal{A}*} = \frac{Nd^2}{2}\frac{(-2\beta+2x)\left(N\Phi^2+\phi^2\right)-N\Phi^2\phi^2}{\left(N\Phi^2+\phi^2\right)\left(-2\beta-N\Phi^2+x\right)\left(-2\beta-\phi^2+x\right)-x\left(N\Phi^2-\phi^2\right)^2}$$

for all permissible x, so we already know that

$$\Pi^{\mathcal{A}*} > \Pi^{\mathcal{P}*} \Longleftrightarrow -\frac{4\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\left(\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}\right)}{2\beta+\frac{N\Phi^2\phi^2}{N\Phi^2+\phi^2}} < x < 0.$$