Online Appendix

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We provide detailed proofs of material that is not included in the main text of our paper "Data-enabled learning, network effects and competitive advantage".

A Additional details for the proof of Proposition 1

Recall that the induction hypothesis is that the result in Proposition 1 and Corollary 1 holds for the states (N_I+1, N_E) and (N_I, N_E+1), i.e. that

$$V^I(N_I{+}1, N_E) = \begin{cases} \frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I+1,N_E) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta(N_I+1, N_E+1) \\ \frac{s_I - s_E + \Delta(N_I+1,N_E)}{(1-\delta)^2} & \text{if} \quad \Delta(N_I+1, N_E+1) \leq s_E - s_I \leq \Delta(N_I+1, N_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta(N_I+1, N_E) \end{cases}$$

$$V^E(N_I, N_E{+}1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(N_I, N_E+1) \\ \frac{s_E - s_I - \Delta(N_I,N_E+1)}{(1-\delta)^2} & \text{if} \quad \Delta(N_I, N_E+1) \leq s_E - s_I \leq \Delta(N_I+1, N_E+1) \\ \frac{s_E - s_I}{1-\delta} - \frac{\Delta(N_I,N_E+1) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I > \Delta(N_I+1, N_E+1) \end{cases}.$$

Also recall from the proof in the main appendix that $V^I(N_I, N_E)$ and $V^E(N_I, N_E)$ can be written (after substituting in the expression for $\Omega(N_I, N_E)$)

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E+1) + \max\left\{ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta \begin{pmatrix} V^I(N_I+1, N_E) + V^E(N_I+1, N_E) \\ -V^I(N_I, N_E+1) - V^E(N_I, N_E+1) \end{pmatrix}, 0 \end{array} \right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I+1, N_E) + \max\left\{ \begin{array}{c} s_E - s_I + f_E(N_E) - f_I(N_I) \\ +\delta \begin{pmatrix} V^E(N_I, N_E+1) + V^I(N_I, N_E+1) \\ -V^E(N_I+1, N_E) - V^I(N_I+1, N_E) \end{pmatrix}, 0 \end{array} \right\}.$$

There are two possibilities. If

$$\max\left\{ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta(V^I(N_I+1, N_E) + V^E(N_I+1, N_E) - V^I(N_I, N_E+1) - V^E(N_I, N_E+1)), 0 \end{array} \right\} > 0,$$

then $V^E(N_I, N_E) = \delta V^E(N_I+1, N_E) = 0$, because we must have $V^E(N_I, N_E) \geq V^E(N_I+1, N_E)$. In this case, we have

$$V^I(N_I, N_E) = s_I - s_E + f_I(N_I) - f_E(N_E) + \delta(V^I(N_I+1, N_E) - V^E(N_I, N_E+1)).$$

On the other hand, if

$$\max\left\{\begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta(V^I(N_I+1,N_E) + V^E(N_I+1,N_E) - V^I(N_I,N_E+1) - V^E(N_I,N_E+1)), 0 \end{array}\right\} = 0,$$

then $V^I(N_I,N_E) = \delta V^I(N_I,N_E+1) = 0$, because we must have $V^I(N_I,N_E) \geq V^I(N_I,N_E+1)$. In this case, we have

$$V^E(N_I,N_E) = s_E - s_I + f_E(N_E) - f_I(N_I) + \delta(V^E(N_I,N_E+1) - V^I(N_I+1,N_E)).$$

Focusing on the first possibility and using the above expressions, we have

$$s_I - s_E + f_I(N_I) - f_E(N_E) + \delta(V^I(N_I+1,N_E) - V^E(N_I,N_E+1))$$

$$= \begin{cases} \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta(\frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I+1,N_E) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2}) \end{array} & \text{if} \quad s_E - s_I < \Delta(N_I,N_E+1) \\ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta(\frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I+1,N_E) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2}) - \delta(\frac{s_E - s_I - \Delta(N_I,N_E+1)}{(1-\delta)^2}) \end{array} & \text{if} \quad \begin{array}{c} \Delta(N_I,N_E+1) \leq s_E - s_I \\ \leq \Delta(N_I+1,N_E+1) \end{array} \\ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta(\frac{s_I - s_E + \Delta(N_I+1,N_E)}{(1-\delta)^2}) - \delta(\frac{s_E - s_I}{1-\delta} - \frac{\Delta(N_I,N_E+1) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2}) \end{array} & \text{if} \quad \begin{array}{c} \Delta(N_I+1,N_E+1) \leq s_E - s_I \\ \leq \Delta(N_I+1,N_E) \end{array} \\ \begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ -\delta(\frac{s_E - s_I}{1-\delta} - \frac{\Delta(N_I,N_E+1) - \delta\Delta(N_I+1,N_E+1)}{(1-\delta)^2}) \end{array} & \text{if} \quad s_E - s_I \geq \Delta(N_I+1,N_E) \end{cases}$$

Straightforward calculations reveal that the expression in the first line is equal to

$$\frac{s_I - s_E}{1-\delta} + \frac{\Delta(N_I,N_E) - \delta\Delta(N_I,N_E+1)}{(1-\delta)^2},$$

while the expressions in the second and third lines are identical and equal to

$$\frac{s_I - s_E + \Delta(N_I,N_E)}{(1-\delta)^2}.$$

Since $s_I - s_E + f_I(N_I) - f_E(N_E) + \delta(V^I(N_I+1,N_E) - V^E(N_I,N_E+1))$ is continuous in $(s_I - s_E)$ and $\Delta(N_I,N_E) < \Delta(N_I+1,N_E)$, we can conclude that the expression of $V^I(N_I,N_E)$ given in Corollary 1 holds. By symmetry in I and E, the same is true for the expression of $V^E(N_I,N_E)$ given in Corollary 1. Thus, the induction hypothesis holds for (N_I,N_E).

B Finite number of periods with pure across-user learning

In this section, we derive the cutoff for the game with pure across-user learning and a finite number of periods. Denote the two firms' profits when there are T periods remaining and the current state is (N_I, N_E) by $\Pi^I(N_I, N_E, T)$ and $\Pi^E(N_I, N_E, T)$.

For $T = 1$ we have

$$\begin{aligned} \Pi^I(N_I, N_E, 1) &= \max\{s_I + f_I(N_I) - s_E - f_E(N_E), 0\} \\ \Pi^E(N_I, N_E, 1) &= \max\{s_E + f_E(N_E) - s_I - f_I(N_I), 0\}, \end{aligned}$$

so

$$\Delta(N_I, N_E, 1) = f_I(N_I) - f_E(N_E).$$

Suppose now that for some $T > 0$ and any state (N_I, N_E) we have

$$\Delta(N_I, N_E, T) = (1-\delta)\left(\sum_{j=0}^{T-1} \frac{\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T + T\delta^{T+1}}\left(f_I(N_I+j) - f_E(N_E+j)\right)\right)$$

$$\Pi^I(N_I, N_E, T) = \begin{cases} \sum_{j=0}^{T-1}\delta^j\left(s_I + f_I(N_I+j) - s_E - f_E(N_E)\right) & \text{if } s_E - s_I < \Delta(N_I, N_E+1, T-1) \\ \left(1+2\delta+\ldots+T\delta^{T-1}\right)\left(s_I - s_E + \Delta(N_I, N_E, T)\right) & \text{if } \Delta(N_I, N_E+1, T-1) \leq s_E - s_I < \Delta(N_I, N_E, T) \\ 0 & \text{if } s_E - s_I \geq \Delta(N_I, N_E, T) \end{cases}$$

$$\Pi^E(N_I, N_E, T) = \begin{cases} 0 & \text{if } s_E - s_I < \Delta(N_I, N_E, T) \\ \left(1+2\delta+\ldots+T\delta^{T-1}\right)\left(s_E - s_I - \Delta(N_I, N_E, T)\right) & \text{if } \Delta(N_I, N_E, T) \leq s_E - s_I < \Delta(N_I+1, N_E, T-1) \\ \sum_{j=0}^{T-1}\delta^j\left(s_E + f_E(N_E+j) - s_I - f_I(N_I)\right) & \text{if } s_E - s_I \geq \Delta(N_I+1, N_E, T-1) \end{cases}.$$

At the end of this section we will confirm that $\Delta(N_I, N_E+1, T-1) < \Delta(N_I, N_E, T) < \Delta(N_I+1, N_E, T-1)$ for any (N_I, N_E) and $T \geq 1$.

Consider the game with $T+1$ periods starting from state (N_I, N_E). I wins iff

$$\begin{aligned} & s_I + f_I(N_I) + \delta\left(\Pi^I(N_I+1, N_E, T) - \Pi^I(N_I, N_E+1, T)\right) \\ > \;& s_E + f_E(N_E) + \delta\left(\Pi^E(N_I, N_E+1, T) - \Pi^E(N_I+1, N_E, T)\right), \end{aligned}$$

so

$$\Pi^I(N_I,N_E,T+1) = \delta\Pi^I(N_I,N_E+1,T)+\max\left\{\begin{array}{c} s_I-s_E+f_I(N_I)-f_E(N_E) \\ +\delta\left(\begin{array}{c}\Pi^I(N_I+1,N_E,T)+\Pi^E(N_I+1,N_E,T)\\ -\Pi^I(N_I,N_E+1,T)-\Pi^E(N_I,N_E+1,T)\end{array}\right),0\end{array}\right\}$$

$$\Pi^E(N_I,N_E,T+1) = \delta\Pi^E(N_I+1,N_E,T)+\max\left\{\begin{array}{c} s_E-s_I+f_E(N_E)-f_I(N_I) \\ +\delta\left(\begin{array}{c}\Pi^I(N_I,N_E+1,T)+\Pi^E(N_I,N_E+1,T)\\ -\Pi^I(N_I+1,N_E,1)-\Pi^E(N_I+1,N_E,T)\end{array}\right),0\end{array}\right\}.$$

Using the induction result for the game with T periods to write the expressions of $\Pi^I(N_I+1,N_E,T)$ and $\Pi^E(N_I,N_E+1,T)$, straightforward calculations yield

$$\Pi^I(N_I,N_E,T+1)=\begin{cases}\sum_{j=0}^{T}\delta^j\left(s_I+f_I(N_I+j)-s_E-f_E(N_E)\right) & \text{if} \quad s_E-s_I<\Delta(N_I,N_E+1,T)\\ \left(1+2\delta+...+(T+1)\delta^T\right)\left(\begin{array}{c}s_I-s_E\\+\Delta(N_I,N_E,T+1)\end{array}\right) & \text{if} \quad \begin{array}{c}\Delta(N_I,N_E+1,T)\le s_E-s_I\\<\Delta(N_I,N_E,T+1)\end{array}\\ 0 & \text{if} \quad s_E-s_I\ge\Delta(N_I,N_E,T+1)\end{cases}$$

$$\Pi^E(N_I,N_E,T+1)=\begin{cases}0 & \text{if} \quad s_E-s_I<\Delta(N_I,N_E,T+1)\\ \left(1+2\delta+...+(T+1)\delta^T\right)\left(\begin{array}{c}s_E-s_I\\-\Delta(N_I,N_E,T+1)\end{array}\right) & \text{if} \quad \begin{array}{c}\Delta(N_I,N_E,T+1)\le s_E-s_I\\<\Delta(N_I+1,N_E,T)\end{array}\\ \sum_{j=0}^{T}\delta^j\left(s_E+f_E(N_E+j)-s_I-f_I(N_I)\right) & \text{if} \quad s_E-s_I\ge\Delta(N_I+1,N_E,T)\end{cases}$$

so

$$\Delta(N_I,N_E,T+1)=(1-\delta)\left(\sum_{j=0}^{T}\frac{\delta^j\left(1-\delta^{T+1-j}\right)}{1-(T+2)\delta^{T+1}+(T+1)\delta^{T+2}}\left(f_I(N_I+j)-f_E(N_E+j)\right)\right)$$

Thus, by induction, the expressions above hold for any $T\ge1$.

Finally, we need to confirm that $\Delta(N_I,N_E+1,T-1)<\Delta(N_I,N_E,T)<\Delta(N_I+1,N_E,T-1)$ for any (N_I,N_E) and $T\ge1$. To do so, write

$$\begin{aligned}\Delta(N_I,N_E,T)-\Delta(N_I,N_E+1,T-1) &= \sum_{j=0}^{T-1}\frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T+T\delta^{T+1}}\left(f_I(N_I+j)-f_E(N_E+j)\right)\\ &\quad-\sum_{j=0}^{T-2}\frac{(1-\delta)\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T}\left(f_I(N_I+j)-f_E(N_E+1+j)\right)\end{aligned}$$

$$
\begin{aligned}
= & \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}} f_{I}\left(N_{I}+j\right)-\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}} f_{I}\left(N_{I}+j\right) \\
& +\sum_{j=0}^{T-2} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}} f_{E}\left(N_{E}+1+j\right)-\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}} f_{E}\left(N_{E}+j\right)
\end{aligned}
$$

It is easily verified that this expression is positive for $T=1$ and $T=2$. So assume $T \geq 3$. Recall that

$$
\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}=\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}=\sum_{j=0}^{T-2} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}=1.
$$

Furthermore, it is straightforward to verify that there exists $j^{*} \in [1, ..., T-2]$ such that

$$
\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}>\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}
$$

for all $0 \leq j \leq j^{*}$ and

$$
\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}} \leq \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}
$$

for all $j^{*}<j \leq T-1$.

Thus, we have

$$
\begin{aligned}
& \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}} f_{I}\left(N_{I}+j\right)-\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}} f_{I}\left(N_{I}+j\right) \\
= & \sum_{j=0}^{j^{*}}\left(\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}\right) f_{I}\left(N_{I}+j\right) \\
& +\sum_{j=j^{*}+1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}\right) f_{I}\left(N_{I}+j\right)
\end{aligned}
$$

$$
\begin{aligned}
> & \sum_{j=0}^{j^{*}}\left(\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}\right) f_{I}\left(N_{I}+j^{*}\right) \\
& +\sum_{j=j^{*}+1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^{T}+T\delta^{T+1}}-\frac{(1-\delta)\,\delta^{j}\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^{T}}\right) f_{I}\left(N_{I}+j^{*}\right) \\
= & \; 0.
\end{aligned}
$$

And

$$\begin{aligned}
&\sum_{j=0}^{T-2} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} f_E\left(N_E+1+j\right) - \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_E\left(N_E+j\right) \\
=\;& \sum_{j=1}^{T-1} \frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} f_E\left(N_E+j\right) - \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_E\left(N_E+j\right)
\end{aligned}$$

$$\begin{aligned}
=\;& \sum_{j=1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} - \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}\right) f_E\left(N_E+j\right) - \frac{(1-\delta)\left(1-\delta^T\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_E\left(N_E\right) \\
>\;& \sum_{j=1}^{T-1}\left(\frac{(1-\delta)\,\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} - \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}}\right) f_E\left(N_E\right) - \frac{(1-\delta)\left(1-\delta^T\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_E\left(N_E\right) \\
=\;& 0,
\end{aligned}$$

where the inequality follows from the observation that $\frac{(1-\delta)\delta^{j-1}\left(1-\delta^{T-j}\right)}{1-T\delta^{T-1}+(T-1)\delta^T} > \frac{(1-\delta)\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\delta^T+T\delta^{T+1}}$ for all j.

We can therefore conclude that

$$\begin{aligned}
&\sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_I\left(N_I+j\right) - \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} f_I\left(N_I+j\right) \\
&+\sum_{j=0}^{T-2} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-1-j}\right)}{1-T\delta^{T-1}+(T-1)\,\delta^T} f_E\left(N_E+1+j\right) - \sum_{j=0}^{T-1} \frac{(1-\delta)\,\delta^j\left(1-\delta^{T-j}\right)}{1-(T+1)\,\delta^T+T\delta^{T+1}} f_E\left(N_E+j\right) \\
>\;& 0,
\end{aligned}$$

so $\Delta\left(N_I, N_E, T\right) > \Delta\left(N_I, N_E+1, T-1\right)$.

The symmetry of I and E then implies that $\Delta\left(N_I, N_E, T\right) < \Delta\left(N_I+1, N_E, T-1\right)$.

Taking the difference,

$$\begin{aligned}
&\Delta(N_I, N_E, T) - \Delta^S(N_I, N_E, T) \\
=\;& (1-\delta)\sum_{j=0}^{T-1} \delta^j\left(\frac{1-\delta^{T-j}}{1-(T+1)\delta^T+T\delta^{T+1}} - \frac{1}{1-\delta^T}\right)\left(f_I(N_I+j) - f_E(N_E+j)\right).
\end{aligned} \tag{B.1}$$

Note that for $T = 1$, which is the case with standard one-period Bertrand competition, there is no distortion, i.e. $\Delta(N_I, N_E, 1) = \Delta^S(N_I, N_E, 1)$. The term in large brackets in (B.1) is positive when $j = 0$, everywhere decreasing in j, and negative when $j = T-1$. Given that $\sum_{j=0}^{T-1}\delta^j\left(\frac{1-\delta^{T-j}}{1-(T+1)\delta^T+T\delta^{T+1}} - \frac{1}{1-\delta^T}\right) = 0$, the overall distortion depends on a weighted average of the difference in the learning functions at consecutive steps, where the weights sum to zero. Thus, in general, for a finite number of periods greater than one, the distortion can go in either direction depending on the shapes of the learning curves.

C Data sharing with across-user learning

We want to show the value functions are given by (15) and (16), where $\Delta(\overline{N}, N_E, \lambda)$ is given by (2). The state $(N_I, N_E) = (\overline{N}, \overline{N})$ is trivial. Recall the value function of profit in this case is given by

$$V^i(\overline{N}, \overline{N}, \lambda) = \frac{\max\{s_i - s_j, 0\}}{1-\delta}.$$

Suppose the value functions hold and the cutoff for E to win the current period given by (2) holds for $(\overline{N}, N_E)$ with $1 \le N_E \le \overline{N}$. We want to show that these also hold for $(\overline{N}, N_E - 1)$. Using the same logic as in the baseline model, we obtain that I wins the current period if and only if $\Omega\left(\overline{N}, N_E - 1, \lambda\right) > 0$, where

$$\Omega\left(\overline{N}, N_E - 1, \lambda\right) = s_I - s_E + f(\overline{N}) - f(N_E - 1) + \delta(1-\lambda)\begin{pmatrix} V^I(\overline{N}, N_E - 1, \lambda) + V^E(\overline{N}, N_E - 1, \lambda) \\ -V^I(\overline{N}, N_E, \lambda) - V^E(\overline{N}, N_E, \lambda) \end{pmatrix}.$$

And the value functions satisfy

$$\begin{aligned} V^I(\overline{N}, N_E - 1, \lambda) &= \delta\left((1-\lambda)V^I(\overline{N}, N_E, \lambda) + \lambda\frac{\max\{s_I - s_E, 0\}}{1-\delta}\right) + \max\left\{\Omega\left(\overline{N}, N_E - 1, \lambda\right), 0\right\} \\ V^E(\overline{N}, N_E - 1, \lambda) &= \delta\left((1-\lambda)V^E(\overline{N}, N_E - 1, \lambda) + \lambda\frac{\max\{s_E - s_I, 0\}}{1-\delta}\right) + \max\left\{-\Omega\left(\overline{N}, N_E - 1, \lambda\right), 0\right\}. \end{aligned}$$

Suppose I wins. In this case, we have

$$\begin{aligned} V^I(\overline{N}, N_E - 1, \lambda) &= \frac{\delta\lambda}{1-\delta(1-\lambda)}\frac{\max\{s_I - s_E, 0\}}{1-\delta} \\ &\quad + \frac{s_I - s_E + f(\overline{N}) - f(N_E - 1) + \delta(1-\lambda)\left(V^E(\overline{N}, N_E - 1, \lambda) - V^E(\overline{N}, N_E, \lambda)\right)}{1-\delta(1-\lambda)} \\ V^E(\overline{N}, N_E - 1, \lambda) &= \frac{\delta\lambda}{1-\delta(1-\lambda)}\frac{\max\{s_E - s_I, 0\}}{1-\delta}. \end{aligned}$$

And from the induction hypothesis, we know

$$V^E(\overline{N}, N_E, \lambda) = \begin{cases} 0 & \text{if} \quad s_E - s_I < 0 \\ \frac{\delta\lambda}{1-\delta(1-\lambda)}\frac{s_E - s_I}{1-\delta} & \text{if} \quad 0 \le s_E - s_I < \Delta\left(\overline{N}, N_E, \lambda\right) \\ \frac{s_E - s_I}{1-\delta} - \frac{\Delta(\overline{N}, N_E, \lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E - s_I \ge \Delta(\overline{N}, N_E, \lambda) \end{cases}.$$

Using these expressions and the identity

$$\Delta(\overline{N}, N_E - 1, \lambda) - \delta(1-\lambda)\Delta\left(\overline{N}, N_E, \lambda\right) = (1 - \delta(1-\lambda))\left(f(\overline{N}) - f(N_E - 1)\right),$$

we obtain

$$V^I(\overline{N}, N_E - 1, \lambda) = \begin{cases} \frac{s_I - s_E}{1-\delta} + \frac{f(\overline{N}) - f(N_E - 1)}{1-\delta(1-\lambda)} & \text{if} \quad s_E - s_I < 0 \\ \frac{s_I - s_E + f(\overline{N}) - f(N_E - 1)}{1-\delta(1-\lambda)} & \text{if} \quad 0 \leq s_E - s_I < \Delta\left(\overline{N}, N_E, \lambda\right) \\ \frac{s_I - s_E + \Delta(\overline{N}, N_E - 1)}{(1-\delta(1-\lambda))^2} & \text{if} \quad \Delta(\overline{N}, N_E, \lambda) \leq s_E - s_I < \Delta(\overline{N}, N_E - 1, \lambda) \end{cases}$$

$$V^E(\overline{N}, N_E - 1, \lambda) = \begin{cases} 0 & \text{if} \quad s_E - s_I < 0 \\ \frac{\delta\lambda}{1-\delta(1-\lambda)} \frac{s_E - s_I}{1-\delta} & \text{if} \quad 0 \leq s_E - s_I < \Delta\left(\overline{N}, N_E - 1, \lambda\right) \end{cases}.$$

This also confirms that $\Delta(\overline{N}, N_E - 1, \lambda)$ is indeed the cutoff such that I wins the current period iff $s_E - s_I < \Delta(\overline{N}, N_E - 1, \lambda)$.

Now suppose E wins, which necessarily means $s_E - s_I > 0$. In this case, we have

$$\begin{aligned} V^I(\overline{N}, N_E - 1, \lambda) &= V^I(\overline{N}, N_E, \lambda) = 0 \\ V^E(\overline{N}, N_E - 1, \lambda) &= \delta\,(1-\lambda)\, V^E(\overline{N}, N_E, \lambda) + \frac{(1 - \delta\,(1-\lambda))\,(s_E - s_I)}{1-\delta} + f(N_E - 1) - f(\overline{N}). \\ &= \frac{s_E - s_I}{1-\delta} - \frac{\Delta(\overline{N}, N_E - 1, \lambda)}{1 - \delta\,(1-\lambda)}. \end{aligned}$$

Combining the above results, shows (15) and (16) hold for $(\overline{N}, N_E - 1)$. By induction, we conclude (15) and (16) hold for all $(\overline{N}, N_E)$, where $0 \leq N_E \leq \overline{N}$.

Finally, we want to show $\Delta(\overline{N}, N_E, \lambda)$ given by (2) is increasing in λ. To shorten expressions, let $\alpha = 1 - \lambda$ so this is equivalent to showing $\Delta(\overline{N}, N_E, 1-\alpha)$ is decreasing in α. We have

$$\frac{d\Delta(\overline{N}, N_E, 1-\alpha)}{d\alpha} = \delta \sum_{j=0}^{\overline{N} - N_E - 1} \left((1-\delta\alpha)\, j - \delta\alpha\right)(\delta\alpha)^{j-1} \left(f(\overline{N}) - f\left(N_E + j\right)\right).$$

There are two cases. If $(1-\delta\alpha)\left(\overline{N} - N_E - 1\right) - \delta\alpha \leq 0$, then the result follows immediately. If instead $(1-\delta\alpha)\left(\overline{N} - N_E - 1\right) - \delta\alpha > 0$, then there exists a unique $j^* \in \left\{0, .., \overline{N} - N_E - 2\right\}$ such that $(1-\delta\alpha)\,j-\delta\alpha<0$ for all $j\leq j^{*}$ and $(1-\delta\alpha)\,j-\delta\alpha\geq 0$ for all $j\geq j^{*}$. In this case, we have

$$\begin{aligned}
\frac{d\Delta(\overline{N},N_E,1-\alpha)}{d\alpha} &= \delta\sum_{j=0}^{\overline{N}-N_E-1}\left((1-\delta\alpha)\,j-\delta\alpha\right)(\delta\alpha)^{j-1}\left(f(\overline{N})-f\left(N_E+j\right)\right)\\
&< \sum_{j=0}^{j^{*}}\left((1-\delta\alpha)\,j-\delta\alpha\right)(\delta\alpha)^{j-1}\left(f(\overline{N})-f\left(N_E+j^{*}\right)\right)\\
&\quad+\sum_{j=j^{*}+1}^{\overline{N}-N_E-1}\left((1-\delta\alpha)\,j-\delta\alpha\right)(\delta\alpha)^{j-1}\left(f(\overline{N})-f\left(N_E+j^{*}\right)\right)\\
&= \left(f(\overline{N})-f\left(N_E+j^{*}\right)\right)\sum_{j=0}^{\overline{N}-N_E-1}\left((1-\delta\alpha)\,j-\delta\alpha\right)(\delta\alpha)^{j-1}\\
&= \left(f(\overline{N})-f\left(N_E+j^{*}\right)\right)\left(\sum_{j=0}^{\overline{N}-N_E-1}j\,(\delta\alpha)^{j-1}-\sum_{j=0}^{\overline{N}-N_E-1}(j+1)\,(\delta\alpha)^{j}\right)<0.
\end{aligned}$$

D Finite number of periods with pure within-user learning

In this section, we prove that the outcome of the game with pure within-user learning and a finite time horizon is socially optimal (unlike the case with across-user learning and a finite time horizon).

Recall that the state (N_I,N_E) here means that the representative consumer has previously purchased N_i times from firm i, for $i=I,E$. If there are t periods remaining and the state is (N_I,N_E), the two firms' profits are denoted $\Pi^{I}\left(N_I,N_E,t\right)$ and $\Pi^{E}\left(N_I,N_E,t\right)$, while the PDV of the net surplus derived by the representative consumer is denoted $u\left(N_I,N_E,t\right)$.

Consider first $t=1$. We have

$$\begin{aligned}
\Pi^{I}\left(N_I,N_E,1\right) &= \max\left\{s_I+f_I\left(N_I\right)-s_E-f_E\left(N_E\right),0\right\}\\
\Pi^{E}\left(N_I,N_E,1\right) &= \max\left\{s_E+f_E\left(N_E\right)-s_I-f_I\left(N_I\right),0\right\}.
\end{aligned}$$

So

$$\begin{aligned}
\Delta\left(N_I,N_E,1\right) &= f_I\left(N_I\right)-f_E\left(N_E\right)\\
u\left(N_I,N_E,1\right) &= \min\left\{s_I-c+f_I\left(N_I\right),s_E-c+f_E\left(N_E\right)\right\}.
\end{aligned}$$

Thus, the result holds for $t=1$.

Now suppose for some $t > 0$, we have

$$\begin{aligned}\Pi^I(N_I, N_E, t) &= \max\left\{\sum_{j=0}^{t-1} \delta^j \left(s_I + f_I(N_I + j) - s_E - f_E(N_E + j)\right), 0\right\} \\ \Pi^E(N_I, N_E, t) &= \max\left\{\sum_{j=0}^{t-1} \delta^j \left(s_E + f_E(N_E + j) - s_I - f_I(N_I + j)\right), 0\right\}\end{aligned}$$

$$\begin{aligned}\Delta(N_I, N_E, t) &= \frac{\sum_{j=0}^{t-1} \delta^j \left(f_I(N_I + j) - f_E(N_E + j)\right)}{\sum_{j=0}^{t-1} \delta^j} \\ u(N_I, N_E, t) &= \min\left\{\sum_{j=0}^{t-1} \delta^j \left(s_I - c + f_I(N_I + j)\right), \sum_{j=0}^{t-1} \delta^j \left(s_E - c + f_E(N_E + j)\right)\right\}\end{aligned}$$

for all (N_I, N_E). Note that $\Delta(N_I, N_E, t)$ defined above is the socially optimum cutoff for the game starting in state (N_I, N_E) and with t periods left.

Consider now the game starting with state (N_I, N_E) and having $t+1$ periods left. I wins iff

$$\begin{aligned}& s_I + f_I(N_I) + \delta\left(\Pi^I(N_I + 1, N_E, t) - \Pi^I(N_I, N_E + 1, t)\right) + \delta u(N_I + 1, N_E, t) \\ > \;& s_E + f_E(N_E) + \delta\left(\Pi^E(N_I, N_E + 1, t) - \Pi^E(N_I + 1, N_E, t)\right) + \delta u(N_I, N_E + 1, t),\end{aligned}$$

so

$$\begin{aligned}\Pi^I(N_I, N_E, t+1) &= \delta \Pi^I(N_I, N_E + 1, t) \\ &\quad + \max\left\{\begin{array}{c} s_I - s_E + f_I(N_I) - f_E(N_E) \\ +\delta\left(\begin{array}{c}\Pi^I(N_I + 1, N_E, t) + \Pi^E(N_I + 1, N_E, t) + u(N_I + 1, N_E, t) \\ -\Pi^I(N_I, N_E + 1, t) - \Pi^E(N_I, N_E + 1, t) - u(N_I, N_E + 1, t)\end{array}\right), 0\end{array}\right\}\end{aligned}$$

$$\begin{aligned}\Pi^E(N_I, N_E, t+1) &= \delta \Pi^E(N_I + 1, N_E, t) \\ &\quad + \max\left\{\begin{array}{c} s_E - s_I + f_E(N_E) - f_I(N_I) \\ +\delta\left(\begin{array}{c}\Pi^I(N_I, N_E + 1, t) + \Pi^E(N_I, N_E + 1, t) + u(N_I, N_E + 1, t) \\ -\Pi^I(N_I + 1, N_E, t) - \Pi^E(N_I + 1, N_E, t) - u(N_I + 1, N_E, t)\end{array}\right), 0\end{array}\right\}.\end{aligned}$$

Using the induction hypothesis for the case with t periods left and states $(N_I + 1, N_E)$ and

(N_I, N_E+1), straightforward calculations lead to

$$\begin{aligned}\Pi^I(N_I,N_E,t+1) &= \max\left\{\sum_{j=0}^{t}\delta^j\left(s_I+f_I(N_I+j)-s_E-f_E(N_E+j)\right),0\right\}\\ \Pi^E(N_I,N_E,t+1) &= \max\left\{\sum_{j=0}^{t}\delta^j\left(s_E+f_E(N_E+j)-s_I-f_I(N_I+j)\right),0\right\}\end{aligned}$$

$$\begin{aligned}\Delta(N_I,N_E,t+1) &= \frac{\sum_{j=0}^{t}\delta^j\left(f_I(N_I+j)-f_E(N_E+j)\right)}{\sum_{j=0}^{t-1}\delta^j}\\ u(N_I,N_E,t+1) &= \min\left\{\sum_{j=0}^{t}\delta^j\left(s_I-c+f_I(N_I+j)\right),\sum_{j=0}^{t}\delta^j\left(s_E-c+f_E(N_E+j)\right)\right\}\end{aligned}$$

Thus, the result holds for $t+1$. By induction, we have thus proven the result for any $t\geq 1$.

E Data sharing with within-user learning

We want to show the value functions and PDV of consumer surplus are given by

$$V^I(\overline{N},N_E,\lambda)=\begin{cases}\frac{s_I-s_E}{1-\delta}+\frac{\Delta(\overline{N},N_E,\lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E-s_I<0\\ \frac{s_I-s_E+\Delta(\overline{N},N_E,\lambda)}{1-\delta(1-\lambda)} & \text{if} \quad 0\leq s_E-s_I<\Delta(\overline{N},N_E,\lambda)\\ 0 & \text{if} \quad s_E-s_I\geq\Delta(\overline{N},N_E,\lambda)\end{cases}$$

$$V^E(\overline{N},N_E,\lambda)=\begin{cases}0 & \text{if} \quad s_E-s_I<0\\ \frac{\delta\lambda}{1-\delta(1-\lambda)}\frac{s_E-s_I}{1-\delta} & \text{if} \quad 0\leq s_E-s_I<\Delta(\overline{N},N_E,\lambda)\\ \frac{s_E-s_I}{1-\delta}-\frac{\Delta(\overline{N},N_E,\lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E-s_I\geq\Delta(\overline{N},N_E,\lambda)\end{cases}$$

$$u(\overline{N},N_E,\lambda)=\begin{cases}\frac{s_E+f(\overline{N})-c}{1-\delta}-\frac{\Delta(\overline{N},N_E,\lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E-s_I<0\\ \frac{s_E+f(\overline{N})-c}{1-\delta}-\frac{\Delta(\overline{N},N_E,\lambda)}{1-\delta(1-\lambda)}+\frac{\delta\lambda}{1-\delta(1-\lambda)}\frac{s_I-s_E}{1-\delta} & \text{if} \quad 0\leq s_E-s_I<\Delta(\overline{N},N_E,\lambda)\\ \frac{s_I+f(\overline{N})-c}{1-\delta} & \text{if} \quad s_E-s_I\geq\Delta(\overline{N},N_E,\lambda)\end{cases},$$

where $\Delta(\overline{N},N_E,\lambda)$ is given by (2).

The case when $(N_I,N_E)=(\overline{N},\overline{N})$ remains the same as in the baseline model, so

$$\begin{aligned}V^I(\overline{N},\overline{N},\lambda) &= \frac{\max\{s_I-s_E,0\}}{1-\delta}\\ V^E(\overline{N},\overline{N},\lambda) &= \frac{\max\{s_E-s_I,0\}}{1-\delta}\end{aligned}$$

$$u(\overline{N},\overline{N},\lambda)=\frac{\min\{s_I,s_E\}+f(\overline{N})-c}{1-\delta}.$$

Consider the case with $N_I=\overline{N}$ and $0\leq N_E<\overline{N}$. Using the same logic as in the baseline model, we obtain that I wins the current period if and only if $\Omega^I\left(\overline{N},N_E,\lambda\right)-\Omega^E\left(\overline{N},N_E,\lambda\right)>0$, where

$$\begin{aligned}
\Omega^I\left(\overline{N},N_E,\lambda\right) &= s_I-c+f(\overline{N})+\delta\left(1-\lambda\right)\left(V^I(\overline{N},N_E,\lambda)-V^I(\overline{N},N_E+1,\lambda)+u(\overline{N},N_E,\lambda)\right)+\delta\lambda u\left(\overline{N},\overline{N},\lambda\right)\\
\Omega^E\left(\overline{N},N_E,\lambda\right) &= s_E-c+f(N_E)+\delta\left(1-\lambda\right)\left(V^E(\overline{N},N_E+1,\lambda)-V^E(\overline{N},N_E,\lambda)+u(\overline{N},N_E+1,\lambda)\right)+\delta\lambda u\left(\overline{N},\right.
\end{aligned}$$

And the value functions satisfy

$$V^I(\overline{N},N_E,\lambda)=\delta\left(\left(1-\lambda\right)V^I(\overline{N},N_E+1,\lambda)+\lambda\frac{\max\{s_I-s_E,0\}}{1-\delta}\right)+\max\left\{\Omega^I\left(\overline{N},N_E,\lambda\right)-\Omega^E\left(\overline{N},N_E,\lambda\right),0\right\}$$

$$V^E(\overline{N},N_E,\lambda)=\delta\left(\left(1-\lambda\right)V^E(\overline{N},N_E,\lambda)+\lambda\frac{\max\{s_E-s_I,0\}}{1-\delta}\right)+\max\left\{\Omega^E\left(\overline{N},N_E,\lambda\right)-\Omega^I\left(\overline{N},N_E,\lambda\right),0\right\}$$

$$u(\overline{N},N_E,\lambda)=\min\left\{\Omega^I\left(\overline{N},N_E,\lambda\right),\Omega^E\left(\overline{N},N_E,\lambda\right)\right\}.$$

Suppose the value functions hold and the cutoff for E to win the current period given by (2) holds for $(\overline{N},N_E+1)$ with $0\leq N_E<\overline{N}$. We want to show that these also hold for $(\overline{N},N_E)$. If I wins, then

$$\begin{aligned}
V^I(\overline{N},N_E,\lambda) &= \frac{\delta\lambda}{1-\delta\left(1-\lambda\right)}\frac{\max\{s_I-s_E,0\}}{1-\delta}+\frac{s_I-s_E+f(\overline{N})-f(N_E)}{1-\delta\left(1-\lambda\right)}\\
&\quad+\frac{\delta\left(1-\lambda\right)}{1-\delta\left(1-\lambda\right)}\left(V^E(\overline{N},N_E,\lambda)+u(\overline{N},N_E,\lambda)-V^E(\overline{N},N_E+1,\lambda)-u(\overline{N},N_E+1,\lambda)\right)
\end{aligned}$$

$$V^E(\overline{N},N_E,\lambda)=\frac{\delta\lambda}{1-\delta\left(1-\lambda\right)}\frac{\max\{s_E-s_I,0\}}{1-\delta}$$

$$u(\overline{N},N_E,\lambda)=s_E-c+f(N_E)+\delta\lambda(V^E(\overline{N},N_E+1,\lambda)+u(\overline{N},N_E+1,\lambda)-V^E(\overline{N},N_E,\lambda))+\delta\lambda u\left(\overline{N},\overline{N},\lambda\right).$$

From the induction hypothesis, we have

$$V^E(\overline{N},N_E+1,\lambda)+u(\overline{N},N_E+1,\lambda)=\frac{s_E+f(\overline{N})-c}{1-\delta}-\frac{\Delta(\overline{N},N_E,\lambda)}{1-\delta\left(1-\lambda\right)}$$

for any s_E-s_I. Using this and the identity

$$\Delta(\overline{N},N_E,\lambda)-\delta\left(1-\lambda\right)\Delta\left(\overline{N},N_E+1,\lambda\right)=\left(1-\delta\left(1-\lambda\right)\right)\left(f(\overline{N})-f(N_E)\right),$$

we obtain

$$V^I(\overline{N}, N_E, \lambda) = \begin{cases} \frac{s_I - s_E}{1-\delta} + \frac{\Delta(\overline{N}, N_E, \lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E - s_I < 0 \\ \frac{s_I - s_E + \Delta(\overline{N}, N_E, \lambda)}{1-\delta(1-\lambda)} & \text{if} \quad 0 \leq s_E - s_I < \Delta(\overline{N}, N_E, \lambda) \end{cases}$$

$$V^E(\overline{N}, N_E, \lambda) = \begin{cases} 0 & \text{if} \quad s_E - s_I < 0 \\ \frac{\delta\lambda}{1-\delta(1-\lambda)} \frac{s_E - s_I}{1-\delta} & \text{if} \quad 0 \leq s_E - s_I < \Delta(\overline{N}, N_E, \lambda) \end{cases}$$

$$u(\overline{N}, N_E, \lambda) = \begin{cases} \frac{s_E + f(\overline{N}) - c}{1-\delta} - \frac{\Delta(\overline{N}, N_E, \lambda)}{1-\delta(1-\lambda)} & \text{if} \quad s_E - s_I < 0 \\ \frac{s_E + f(\overline{N}) - c}{1-\delta} - \frac{\Delta(\overline{N}, N_E, \lambda)}{1-\delta(1-\lambda)} + \frac{\delta\lambda}{1-\delta(1-\lambda)} \frac{s_I - s_E}{1-\delta} & \text{if} \quad 0 \leq s_E - s_I < \Delta(\overline{N}, N_E, \lambda) \end{cases}.$$

This also confirms that $\Delta(\overline{N}, N_E, \lambda)$ is indeed the cutoff such that I wins the current period iff $s_E - s_I < \Delta(\overline{N}, N_E, \lambda)$.

Now suppose E wins, which necessarily means $s_E - s_I > 0$. In this case, we have

$$V^I(\overline{N}, N_E, \lambda) = V^I(\overline{N}, N_E + 1, \lambda) = 0$$

$$\begin{aligned} V^E(\overline{N}, N_E, \lambda) &= \delta\lambda \frac{\max\{s_E - s_I, 0\}}{1-\delta} + s_E - s_I + f(N_E) - f(\overline{N}) \\ &\quad + \delta(1-\lambda)\left(V^E(\overline{N}, N_E + 1, \lambda) + u(\overline{N}, N_E + 1, \lambda) - u(\overline{N}, N_E, \lambda)\right) \end{aligned}$$

$$u(\overline{N}, N_E, \lambda) = \frac{s_I - c + f(\overline{N})}{1 - \delta(1-\lambda)} + \frac{\delta\lambda}{1-\delta(1-\lambda)} u\left(\overline{N}, \overline{N}\right).$$

Once again using the induction hypothesis and the identity above linking $\Delta(\overline{N}, N_E, \lambda)$ and $\Delta(\overline{N}, N_E + 1, \lambda)$, we obtain

$$V^I(\overline{N}, N_E, \lambda) = 0$$

$$V^E(\overline{N}, N_E, \lambda) = \frac{s_E - s_I}{1-\delta} - \frac{\Delta(\overline{N}, N_E)}{1-\delta(1-\lambda)}$$

$$u(\overline{N}, N_E, \lambda) = \frac{s_I + f(\overline{N}) - c}{1-\delta}.$$

So the result also holds for $\left(\overline{N}, N_E\right)$. By induction, we conclude the result holds for all $(\overline{N}, N_E)$, where $0 \leq N_E \leq \overline{N}$.

F Proof of Proposition 12

We first treat the case with Pareto beliefs. In addition to proving that the cutoff $\Delta(N_I, N_E)$ takes the expression stated in Proposition 12 in the main text, we want to show the two firms' value functions are

$$V^I(N_I,N_E)=\begin{cases}\frac{s_I-s_E}{(1-\delta)^2}+\frac{\Delta(N_I,N_E)-\delta\Delta(N_I,N_E+1)}{(1-\delta)^3} & \text{if} \quad s_E-s_I<\Delta(N_I,N_E+1)\\ \frac{s_I-s_E+\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(N_I,N_E+1)\leq s_E-s_I<\Delta(N_I,N_E)\\ 0 & \text{if} \quad s_E-s_I\geq\Delta(N_I,N_E)\end{cases}$$

$$V^E(N_I,N_E)=\begin{cases}0 & \text{if} \quad s_E-s_I<\Delta(N_I,N_E)\\ \frac{s_E-s_I-\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(N_I,N_E)\leq s_E-s_I<\Delta(N_I+1,N_E)\\ \frac{s_E-s_I}{(1-\delta)^2}-\frac{\Delta(N_I,N_E)-\delta\Delta(N_I+1,N_E)}{(1-\delta)^3} & \text{if} \quad s_E-s_I\geq\Delta(N_I+1,N_E)\end{cases}.$$

Assume initially that consumers would never consider buying both products, an assumption we will relax at the end. Suppose to start with $N_I=\overline{N}_I$ and $N_E=\overline{N}_E$, i.e. both firms have reached their respective thresholds, in which case beliefs do not matter. Suppose I charges p^I and E charges p^E in the current period. Then because consumers care about the PDV of their stream of future utility, new consumers in the current period choose I iff

$$\frac{s_I+f_I(\overline{N}_I)}{1-\delta}-p^I>\frac{s_E+f_E(\overline{N}_E)}{1-\delta}-p^E.$$

Furthermore, since learning has been exhausted for both firms, there is no future benefit to a firm of attracting the current consumers, beyond the price that it collects. Thus, both I and E are only willing to price as low as c in order to win in the current period. Given Bertrand competition and Pareto beliefs, and the fact firms can sell to a new set of consumers every period in the same way, we have

$$\begin{aligned}V^I(\overline{N}_I,\overline{N}_E) &= \max\left\{\frac{s_I-s_E+f_I(\overline{N}_I)-f_E(\overline{N}_E)}{(1-\delta)^2},0\right\}\\ V^E(\overline{N}_I,\overline{N}_E) &= \max\left\{\frac{s_E-s_I+f_E(\overline{N}_E)-f_I(\overline{N}_I)}{(1-\delta)^2},0\right\},\end{aligned}$$

which implies $\Delta(\overline{N}_I,\overline{N}_E)=f_I(\overline{N}_I)-f_E(\overline{N}_E)$.

Suppose the state is $(\overline{N}_I,N_E)$ where $1\leq N_E\leq\overline{N}_E$.[1] We have

$$V^I(\overline{N}_I,N_E)=\begin{cases}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^2} & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E+1)\\ \frac{s_I-s_E+\Delta(\overline{N}_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta(\overline{N}_I,N_E+1)\leq s_E-s_I <\Delta(\overline{N}_I,N_E)\\ 0 & \text{if} \quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

$$V^E(\overline{N}_I,N_E)=\begin{cases}0 & \text{if} \quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_E-s_I-\Delta(\overline{N}_I,N_E)}{(1-\delta)^2} & \text{if} \quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

[1] The expressions below go through for $N_E=\overline{N}_E$ with some slight abuse of notation.

$$\Delta(\overline{N}_I,N_E)=f_I(\overline{N}_I)-(1-\delta)^2\sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j)-\delta^{\overline{N}_E-N_E}(\overline{N}_E-N_E+1-(\overline{N}_E-N_E)\delta)f_E(\overline{N}_E).$$

Consider now the state $(\overline{N}_I, N_E-1)$. Using the generalization of (**??**) from the main text to the current setting in which consumers get a lifetime of utility from a purchase decision, I wins iff

$$\begin{aligned}&\frac{s_I+f_I(\overline{N}_I)}{1-\delta}+\delta(V^I(\overline{N}_I,N_E-1)-V^I(\overline{N}_I,N_E))\\ >&\sum_{j=0}^{\overline{N}_E-N_E}\delta^j(s_E+f_E(N_E-1+j))+\delta^{\overline{N}_E-N_E+1}(\frac{s_E+f_E(\overline{N}_E)}{1-\delta})+\delta(V^E(\overline{N}_I,N_E)-V^E(\overline{N}_I,N_E-1)).\end{aligned}$$

Following the same logic as in the derivation of (11)-(12) in the main text, this implies

$$V^I(\overline{N}_I,N_E-1)=\delta V^I(\overline{N}_I,N_E)+\max\left\{\begin{array}{c}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{1-\delta}\\ +\delta\left(\begin{array}{c}V^I(\overline{N}_I,N_E-1)+V^E(\overline{N}_I,N_E-1)\\ -V^I(\overline{N}_I,N_E)-V^E(\overline{N}_I,N_E)\end{array}\right)\end{array},0\right\}$$

$$V^E(\overline{N}_I,N_E-1)=\delta V^E(\overline{N}_I,N_E-1)+\max\left\{\begin{array}{c}\frac{s_E+(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)+\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)-s_I-f_I(\overline{N}_I)}{1-\delta}\\ +\delta\left(\begin{array}{c}V^I(\overline{N}_I,N_E)+V^E(\overline{N}_I,N_E)\\ -V^I(\overline{N}_I,N_E-1)-V^E(\overline{N}_I,N_E-1)\end{array}\right)\end{array},0\right\}.$$

There are two cases. Suppose first $V^E(\overline{N}_I, N_E-1)=0$. Following the same steps as in the proof of Proposition 1 in the main text, we have

$$\begin{aligned}V^I(\overline{N}_I,N_E-1)&=\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2}\\ &\quad-\frac{\delta}{1-\delta}V^E(\overline{N}_I,N_E)\end{aligned}$$

$$=\begin{cases}\frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2} & \text{if}\quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_I+f_I(\overline{N}_I)-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{(1-\delta)^2}-\frac{\delta}{1-\delta}\frac{s_E-s_I-\Delta(\overline{N}_I,N_E)}{(1-\delta)^2} & \text{if}\quad s_E-s_I\geq\Delta(\overline{N}_I,N_E)\end{cases}$$

thus establishing that

$$V^I(\overline{N}_I,N_E-1)=\begin{cases}\frac{s_I-s_E}{(1-\delta)^2}+\frac{\Delta(\overline{N}_I,N_E-1)-\delta\Delta(\overline{N}_I,N_E)}{(1-\delta)^3} & \text{if}\quad s_E-s_I<\Delta(\overline{N}_I,N_E)\\ \frac{s_I-s_E+\Delta(\overline{N}_I,N_E-1)}{(1-\delta)^3} & \text{if}\quad \Delta(\overline{N}_I,N_E)\leq s_E-s_I\leq\Delta(\overline{N}_I,N_E-1)\\ 0 & \text{if}\quad s_E-s_I\geq\Delta(\overline{N}_I,N_E-1)\end{cases},$$

where $\Delta(\overline{N}_I, N_E-1)$ is consistent with the general formula given in the proposition. Similarly, following the steps of Proposition 1 in the main text, if $V^E(\overline{N}_I, N_E-1) > 0$ we obtain

$$V^I(\overline{N}_I, N_E-1) = V^I(\overline{N}_I, N_E) = 0$$

$$\begin{aligned} V^E(\overline{N}_I, N_E-1) &= \frac{s_E + (1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j) + \delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E) - s_I - f_I(\overline{N}_I)}{1-\delta} \\ &\quad +\delta V^E(\overline{N}_I, N_E) \end{aligned}$$

$$= \begin{cases} \frac{s_E - s_I - f_I(\overline{N}_I) + (1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j) + \delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)}{1-\delta} & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E) \\ \frac{\substack{s_E - s_I - f_I(\overline{N}_I) + (1-\delta)^2\sum_{j=0}^{\overline{N}_E-N_E}(j+1)\delta^j f_E(N_E-1+j) \\ +\delta^{\overline{N}_E-N_E+1}(\overline{N}_E - N_E + 2 - (\overline{N}_E - N_E+1)\delta)f_E(\overline{N}_E)}}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E) \end{cases},$$

and so

$$V^E(\overline{N}_I, N_E-1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(\overline{N}_I, N_E-1) \\ \frac{s_E - s_I - \Delta(\overline{N}_I, N_E-1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta(\overline{N}_I, N_E-1) \end{cases}.$$

Thus, combining the two cases, we conclude that the result holds for the state $(\overline{N}_I, N_E-1)$. By induction, this implies it holds for all states $(N_I = \overline{N}_I, N_E)$ with $0 \leq N_E \leq \overline{N}_E$. By symmetry, it also holds for any state $(N_I, N_E = \overline{N}_E)$ with $0 \leq N_I \leq \overline{N}_I$ with $m \geq 0$.

Consider now the case (N_I, N_E) with $0 \leq N_I \leq \overline{N}_I - 1$ and $0 \leq N_E \leq \overline{N}_E - 1$, and suppose result holds for (N_I+1, N_E) and (N_I, N_E+1). I wins the current period iff

$$\begin{aligned} &\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j(s_I + f_I(N_I+j)) + \delta^{\overline{N}_I-N_I}(\frac{s_I + f_I(\overline{N}_I)}{1-\delta}) + \delta(V^I(N_I+1, N_E) - V^I(N_I, N_E+1)) \\ > &\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j(s_E + f_E(N_E+j)) + \delta^{\overline{N}_E-N_E}(\frac{s_E + f_E(\overline{N}_E)}{1-\delta}) + \delta(V^E(N_I, N_E+1) - V^E(N_I+1, N_E)). \end{aligned}$$

We then have

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E+1) + \max\left\{ \begin{array}{c} \frac{s_I - s_E + \delta^{\overline{N}_I-N_I}f_I(\overline{N}_I) - \delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{1-\delta} \\ +\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j) - \sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j) \\ +\delta\left(\begin{array}{c} V^I(N_I+1, N_E) + V^E(N_I+1, N_E) \\ -V^I(N_I, N_E+1) - V^E(N_I, N_E+1) \end{array}\right), 0 \end{array}\right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I+1, N_E) + \max\left\{ \begin{array}{c} \frac{s_E - s_I + \delta^{\overline{N}_E-N_E}f_E(\overline{N}_E) - \delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{1-\delta} \\ +\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j) - \sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j(f_I(N_I+j)) \\ +\delta\left(\begin{array}{c} V^I(N_I, N_E+1) + V^E(N_I, N_E+1) \\ -V^I(N_I+1, N_E) - V^E(N_I+1, N_E) \end{array}\right), 0 \end{array}\right\}.$$

Two cases. Suppose first I wins the current period, so

$$V^E(N_I, N_E) = V^E(N_I+1, N_E) = 0$$

and

$$\begin{aligned} V^I(N_I, N_E) &= \frac{s_I - s_E + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{1-\delta} \\ &\quad + \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) \\ &\quad + \delta(V^I(N_I+1, N_E) - V^E(N_I, N_E+1)) \end{aligned}$$

From the induction hypothesis, we have

$$V^I(N_I{+}1, N_E) = \begin{cases} \begin{array}{c} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^2} \\ + \sum_{j=0}^{\overline{N}_I - N_I - 2} (j+1)\delta^j f_I(N_I + 1 + j) \\ - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j)}{1-\delta} \end{array} & \text{if } \begin{array}{c} s_E - s_I \\ < \Delta(N_I+1, N_E+1) \end{array} \\ \frac{s_I - s_E + \Delta(N_I+1, N_E)}{(1-\delta)^3} & \text{if } \begin{array}{c} \Delta(N_I+1, N_E+1) \\ \leq s_E - s_I \\ < \Delta(N_I+1, N_E) \end{array} \\ 0 & \text{if } s_E - s_I \geq \Delta(N_I+1, N_E) \end{cases}$$

$$V^E(N_I, N_E{+}1) = \begin{cases} 0 & \text{if } s_E - s_I < \Delta(N_I, N_E+1) \\ \frac{s_E - s_I - \Delta(N_I, N_E+1)}{(1-\delta)^3} & \text{if } \begin{array}{c} \Delta(N_I, N_E+1) \\ \leq s_E - s_I \\ < \Delta(N_I+1, N_E+1) \end{array} \\ \begin{array}{c} \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1}(\overline{N}_E - N_E - (\overline{N}_E - N_E - 1)\delta) f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ + \sum_{j=0}^{\overline{N}_E - N_E - 2} (j+1)\delta^j f_E(N_E + 1 + j) \\ - \frac{\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} \end{array} & \text{if } \begin{array}{c} s_E - s_I \\ \geq \Delta(N_I+1, N_E+1) \end{array} \end{cases}$$

Plugging these two expressions into the expression of $V^I(N_I, N_E)$ above, we obtain

$$V^I(N_I, N_E) = \frac{s_I - s_E + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{1-\delta} + \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I{+}j) - \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E{+}j)$$

$$
+\delta\begin{cases}
\frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I - N_I - 2}(j+1)\delta^j f_I(N_I+1+j) - \frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{1-\delta} & \text{if } s_E - s_I < \Delta(N_I+1, N_E+1) \\
\frac{s_I - s_E + \delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I) - \delta^{\overline{N}_E-N_E}(\overline{N}_E-N_E+1-(\overline{N}_E-N_E)\delta)f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j) - \sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j)}{1-\delta} & \text{if } \Delta(N_I+1,N_E+1) \leq s_E-s_I < \Delta(N_I+1,N_E) \\
0 & \text{if } s_E - s_I \geq \Delta(N_I+1,N_E)
\end{cases}
$$

$$
-\delta\begin{cases}
0 & \text{if } s_E-s_I < \Delta(N_I, N_E+1) \\
\frac{s_E - s_I - \delta^{\overline{N}_I-N_I}(\overline{N}_I-N_I+1-(\overline{N}_I-N_I)\delta)f_I(\overline{N}_I) + \delta^{\overline{N}_E-N_E-1}(\overline{N}_E-N_E-(\overline{N}_E-N_E-1)\delta)f_E(\overline{N}_E)}{(1-\delta)^3} + \frac{\sum_{j=0}^{\overline{N}_E-N_E-2}(j+1)\delta^j f_E(N_E+1+j) - \sum_{j=0}^{\overline{N}_I-N_I-1}(j+1)\delta^j f_I(N_I+j)}{1-\delta} & \text{if } \Delta(N_I,N_E+1) \leq s_E-s_I < \Delta(N_I+1,N_E+1) \\
\frac{s_E - s_I + \delta^{\overline{N}_E-N_E-1}(\overline{N}_E-N_E-(\overline{N}_E-N_E-1)\delta)f_E(\overline{N}_E) - \delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_E-N_E-2}(j+1)\delta^j f_E(N_E+1+j) - \frac{\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if } s_E-s_I \geq \Delta(N_I+1,N_E+1)
\end{cases}
$$

Straightforward calculations and the imposition of the condition $V^I(N_I, N_E) \geq 0$ lead to

$$
V^I(N_I,N_E) = \begin{cases}
\frac{s_I-s_E+\delta^{\overline{N}_I-N_I}(\overline{N}_I-N_I+1-(\overline{N}_I-N_I)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^2} + \sum_{j=0}^{\overline{N}_I-N_I-1}(j+1)\delta^j f_I(N_I+j) - \frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{1-\delta} & \text{if } s_E - s_I < \Delta(N_I, N_E+1) \\
\frac{s_I-s_E+\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if } \Delta(N_I,N_E+1) \leq s_E-s_I < \Delta(N_I,N_E) \\
0 & \text{if } s_E - s_I \geq \Delta(N_I,N_E)
\end{cases},
$$

where

$$\Delta(N_I,N_E) = (1-\delta)^2 \sum_{j=0}^{\overline{N}_I-N_I-1} (j+1)\delta^j f_I(N_I+j) + \delta^{\overline{N}_I-N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I)$$
$$-(1-\delta)^2 \sum_{j=0}^{\overline{N}_E-N_E-1} (j+1)\delta^j f_E(N_E+j) - \delta^{\overline{N}_E-N_E}(\overline{N}_E - N_E + 1 - (\overline{N}_E - N_E)\delta) f_E(\overline{N}_E).$$

The case in which E wins the current period (so $V^I(N_I,N_E) = V^I(N_I,N_E+1) = 0$) is symmetric, so we have $V^E(N_I,N_E) = V^E(N_I+1,N_E) = 0$ and

$$V^E(N_I,N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta(N_I,N_E) \\ \frac{s_E-s_I-\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c}\Delta(N_I,N_E) \le s_E - s_I \\ < \Delta(N_I+1,N_E)\end{array} \\ \begin{array}{c}\frac{s_E-s_I+\delta^{\overline{N}_E-N_E}(\overline{N}_E-N_E+1-(\overline{N}_E-N_E)\delta)f_E(\overline{N}_E)-\delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} \\ +\sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j) \\ -\frac{\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta}\end{array} & \text{if} \quad s_E - s_I \ge \Delta(N_I+1,N_E) \end{cases}$$

Thus, the result holds for the state (N_I,N_E). Using this induction reasoning repeatedly, we conclude that the result holds for every state (N_I,N_E) with $0 \le N_I \le \overline{N}_I$ and $0 \le N_E \le \overline{N}_E$.

Finally, along the equilibrium path we have just determined, in any given period, "old" consumers (i.e. those that arrived in previous periods) do not want to switch and also purchase from the losing firm. This is because consumers that arrive in the current period do not purchase from the losing firm in equilibrium. Indeed, for any given prices charged by the two firms (including equilibrium prices), old consumers are less willing to buy from the losing firm than current consumers, because the former pay nothing to keep consuming from the winning firm. The only exception to this logic is if prices are ever negative, which we ruled out by assumption in the setup of our model.

To see that the outcome with Pareto beliefs is socially efficient, compare the total PDVs of utility created by the two firms for all current and future consumers. E creates more total utility iff

$$\sum_{j=0}^{\overline{N}_E-N_E-1} \delta^j \Big(\sum_{k=0}^{\overline{N}_E-N_E-1-j} \delta^k (s_E + f_E(N_E+j+k)) + \delta^{\overline{N}_E-N_E-j}\frac{s_E+f_E(\overline{N}_E)}{1-\delta}\Big) + \delta^{\overline{N}_E-N_E}\frac{s_E+f_E(\overline{N}_E)}{(1-\delta)^2}$$
$$\ge \sum_{j=0}^{\overline{N}_I-N_I-1} \delta^j \Big(\sum_{k=0}^{\overline{N}_I-N_I-1-j} \delta^k (s_I + f_I(N_I+j+k)) + \delta^{\overline{N}_I-N_I-j}\frac{s_I+f_I(\overline{N}_I)}{1-\delta}\Big) + \delta^{\overline{N}_I-N_I}\frac{s_I+f_I(\overline{N}_I)}{(1-\delta)^2}.$$

Simple calculations show that this is equivalent to

$$\frac{s_E + (1-\delta)^2 \sum_{j=0}^{\overline{N}_E - N_E - 1} (j+1)\delta^j f_E(N_E + j) + \delta^{\overline{N}_E - N_E}(\overline{N}_E - N_E + 1 - (\overline{N}_E - N_E)\delta) f_E(\overline{N}_E)}{(1-\delta)^2}$$
$$\geq \quad \frac{s_I + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I)}{(1-\delta)^2}$$

which in turn is equivalent to $s_E - s_I \geq \Delta(N_I, N_E)$.

Consider now the case with beliefs that favor I. In this case, in addition to establishing the cutoff $\Delta^I(N_I, N_E)$ defined in the proposition, we want to show the two firms' value functions are

$$V^I(N_I, N_E) = \begin{cases} \frac{s_I - s_E}{(1-\delta)^2} + \frac{\Delta^I(N_I, N_E) - \delta\Delta^I(N_I, N_E+1)}{(1-\delta)^3} & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E + 1) \\ \frac{s_I - s_E + \Delta^I(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E + 1) \leq s_E - s_I < \Delta^I(N_I, N_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, N_E) \end{cases},$$

$$V^E(N_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E) \\ \frac{s_E - s_I - \Delta(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E) \leq s_E - s_I < \Delta^I(N_I + 1, N_E) \\ \frac{s_E - s_I}{(1-\delta)^2} - \frac{\Delta^I(N_I, N_E) - \delta\Delta^I(N_I+1, N_E)}{(1-\delta)^3} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I + 1, N_E) \end{cases}.$$

Nothing changes for the state $(N_I, N_E) = (\overline{N}_I, \overline{N}_E)$, so $\Delta^I(\overline{N}_I, \overline{N}_E) = f_I(\overline{N}_I) - f_E(\overline{N}_E)$ and $V^I(\overline{N}_I, \overline{N}_E)$ and $V^E(\overline{N}_I, \overline{N}_E)$ are defined as above. Suppose the result holds for the state $(\overline{N}_I, N_E)$ with $1 \leq N_E \leq \overline{N}_E$, i.e.

$$\Delta^I(\overline{N}_I, N_E) = f_I(\overline{N}_I) - (1-\delta) \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)$$

$$V^I(\overline{N}_I, N_E) = \begin{cases} \frac{s_I - s_E + f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E + 1) \\ \frac{s_I - s_E + f_I(\overline{N}_I) - (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c} \Delta^I(\overline{N}_I, N_E + 1) \leq s_E - s_I \\ < \Delta^I(\overline{N}_I, N_E) \end{array} \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}$$

$$V^E(\overline{N}_I, N_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_E - s_I + (1-\delta)\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) + \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E) - f_I(\overline{N}_I)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E) \end{cases}.$$

Now consider the state $(\overline{N}_I, N_E - 1)$. Favorable beliefs for I imply I wins the current period iff

$$\frac{s_I + f_I(\overline{N}_I)}{1-\delta} + \delta(V^I(\overline{N}_I, N_E - 1) - V^I(\overline{N}_I, N_E)) > \frac{s_E + f_E(N_E - 1)}{1-\delta} + \delta(V^E(\overline{N}_I, N_E) - V^E(\overline{N}_I, N_E - 1)).$$

And following the same steps as above, we have

$$V^I(\overline{N}_I, N_E-1) = \delta V^I(\overline{N}_I, N_E) + \max\left\{ \begin{array}{c} \frac{s_I+f_I(\overline{N}_I)-s_E-f_E(N_E-1)}{1-\delta} \\ +\delta\left(\begin{array}{c} V^I(\overline{N}_I, N_E-1)+V^E(\overline{N}_I, N_E-1) \\ -V^I(\overline{N}_I, N_E)-V^E(\overline{N}_I, N_E) \end{array}\right), 0 \end{array} \right\}$$

$$V^E(\overline{N}_I, N_E-1) = \delta V^E(\overline{N}_I, N_E-1) + \max\left\{ \begin{array}{c} \frac{s_E+f_E(N_E-1)-s_I-f_I(\overline{N}_I)}{1-\delta} \\ +\delta\left(\begin{array}{c} V^I(\overline{N}_I, N_E)+V^E(\overline{N}_I, N_E) \\ -V^I(\overline{N}_I, N_E-1)-V^E(\overline{N}_I, N_E-1) \end{array}\right), 0 \end{array} \right\}.$$

There are two possibilities: $V^E(\overline{N}_I, N_E-1) = 0$ and $V^E(\overline{N}_I, N_E-1) > 0$.

Suppose first $V^E(\overline{N}_I, N_E-1) = 0$, so $V^I(\overline{N}_I, N_E-1) \geq \delta V^I(\overline{N}_I, N_E)$ and

$$\begin{aligned} V^I(\overline{N}_I, N_E-1) &= \frac{s_I+f_I(\overline{N}_I)-s_E-f_E(N_E-1)}{(1-\delta)^2} - \frac{\delta}{1-\delta}V^E(\overline{N}_I, N_E) \\ &= \left\{ \begin{array}{cl} \frac{s_I-s_E+f_I(\overline{N}_I)-f_E(N_E-1)}{(1-\delta)^2} & \text{if} \quad s_E-s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_I-s_E-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E}\delta^j f_E(N_E-1+j)-\delta^{\overline{N}_E-N_E+1}f_E(\overline{N}_E)+f_I(\overline{N}_I)}{(1-\delta)^3} & \text{if} \quad s_E-s_I \geq \Delta^I(\overline{N}_I, N_E) \end{array} \right. . \end{aligned}$$

Note that when $s_E - s_I < \Delta^I(\overline{N}_I, N_E)$, we have

$$\frac{s_I-s_E+f_I(\overline{N}_I)-f_E(N_E-1)}{(1-\delta)^2} > \frac{s_I-s_E+f_I(\overline{N}_I)-f_E(N_E)}{(1-\delta)^2} = V^I(\overline{N}_I, N_E) > \delta V^I(\overline{N}_I, N_E)$$

and when $s_E - s_I \geq \Delta^I(\overline{N}_I, N_E)$, we have $V^I(\overline{N}_I, N_E) = 0$, so the binding constraint on this range must be $V^I(\overline{N}_I, N_E-1) \geq 0$, which is equivalent to $s_E - s_I \leq \Delta^I(\overline{N}_I, N_E-1)$.

Next, suppose $V^E(\overline{N}_I, N_E-1) > 0$, so $V^I(\overline{N}_I, N_E-1) = \delta V^I(\overline{N}_I, N_E)$. Since $V^I(\overline{N}_I, N_E-1) \geq V^I(\overline{N}_I, N_E)$ and $\delta < 1$, this implies $V^I(\overline{N}_I, N_E-1) = V^I(\overline{N}_I, N_E) = 0$, so

$$\begin{aligned} V^E(\overline{N}_I, N_E-1) &= \frac{s_E+\delta f_E(N_E-1)-s_I-f_I(\overline{N}_I)}{1-\delta} + \delta V^E(\overline{N}_I, N_E) \\ &= \left\{ \begin{array}{cl} \frac{s_E-s_I+f_E(N_E-1)-f_I(\overline{N}_I)}{1-\delta} & \text{if} \quad s_E-s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_E-s_I-\Delta^I(\overline{N}_I, N_E-1)}{(1-\delta)^2} & \text{if} \quad s_E-s_I \geq \Delta^I(\overline{N}_I, N_E) \end{array} \right. . \end{aligned}$$

Note that $s_E - s_I + f_E(N_E-1) - f_I(\overline{N}_I) < s_E - s_I + f_E(N_E) - f_I(\overline{N}_I) < 0$ given $s_E - s_I < \Delta^I(\overline{N}_I, N_E)$, and $s_E - s_I - \Delta^I(\overline{N}_I, N_E-1) > 0$ iff $s_E - s_I \geq \Delta^I(\overline{N}_I, N_E-1)$.

Thus, we have proven

$$V^I(\overline{N}_I, N_E-1) = \left\{ \begin{array}{ccc} \frac{s_I-s_E+f_I(\overline{N}_I)-f_E(N_E-1)}{(1-\delta)^2} & \text{if} & s_E-s_I < \Delta^I(\overline{N}_I, N_E) \\ \frac{s_I-s_E+\Delta^I(\overline{N}_I, N_E-1)}{(1-\delta)^3} & \text{if} & \Delta^I(\overline{N}_I, N_E) \leq s_E-s_I < \Delta^I(\overline{N}_I, N_E-1) \\ 0 & \text{if} & s_E-s_I \geq \Delta^I(\overline{N}_I, N_E-1) \end{array} \right. .$$

$$V^E(\overline{N}_I, N_E-1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(\overline{N}_I, N_E-1) \\ \frac{s_E - s_I - \Delta^I(\overline{N}_I, N_E-1)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(\overline{N}_I, N_E-1) \end{cases}.$$

So the result holds for the state $(\overline{N}_I, N_E - 1)$. By repeated application of the induction hypothesis, the result holds for all states $(\overline{N}_I, N_E)$, with $0 \leq N_E \leq \overline{N}_E$.

For any state $(N_I, \overline{N}_E)$ with $0 \leq N_I \leq \overline{N}_I$, the outcome is the same as under Pareto expectations because E is already at its threshold, so expectations don't affect it. This means we already know we have

$$\Delta(N_I, \overline{N}_E) = \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(\overline{N}_E) + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j)$$

$$V^I(N_I, \overline{N}_E) = \begin{cases} \dfrac{s_I - s_E + (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1}(j+1)\delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(\overline{N}_E)}{(1-\delta)^2} & \text{if} \quad s_E - s_I < \Delta^I(N_I, \overline{N}_E) \\ 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, \overline{N}_E) \end{cases}$$

$$V^E(N_I, \overline{N}_E) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, \overline{N}_E) \\ \frac{s_E - s_I - \Delta(N_I, \overline{N}_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, \overline{N}_E) \leq s_E - s_I < \Delta^I(N_I + 1, \overline{N}_E) \\ \frac{s_E - s_I + f_E(\overline{N}_E) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I + 1, \overline{N}_E) \end{cases},$$

which means the result holds for all states $(N_I, \overline{N}_E)$, with $0 \leq N_I \leq \overline{N}_I$.

Consider now the state (N_I, N_E) with $0 \leq N_I \leq \overline{N}_I - 1$ and $0 \leq N_E \leq \overline{N}_E - 1$, and suppose the result we want to prove holds for states $(N_I + 1, N_E)$ and $(N_I, N_E + 1)$. I wins the current period iff

$$\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (s_I + f_I(N_I + j)) + \delta^{\overline{N}_I - N_I}\left(\frac{s_I + f_I(\overline{N}_I)}{1-\delta}\right) + \delta(V^I(N_I + 1, N_E) - V^I(N_I, N_E + 1))$$
$$> \frac{s_E + f_E(N_E)}{1-\delta} + \delta(V^E(N_I, N_E + 1) - V^E(N_I + 1, N_E)).$$

We then have

$$V^I(N_I, N_E) = \delta V^I(N_I, N_E + 1) + \max\left\{ \begin{array}{c} \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(N_I + 1, N_E) + V^E(N_I + 1, N_E) \\ -V^I(N_I, N_E + 1) - V^E(N_I, N_E + 1) \end{array} \right), 0 \end{array} \right\}$$

$$V^E(N_I, N_E) = \delta V^E(N_I + 1, N_E) + \max\left\{ \begin{array}{c} \frac{s_E - s_I + f_E(N_E) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j (f_I(N_I + j)) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{1-\delta} \\ +\delta \left(\begin{array}{c} V^I(N_I, N_E + 1) + V^E(N_I, N_E + 1) \\ -V^I(N_I + 1, N_E) - V^E(N_I + 1, N_E) \end{array} \right), 0 \end{array} \right\}.$$

There are two cases. Suppose first I wins the current period, so $V^E(N_I, N_E) = V^E(N_I + 1, N_E) = 0$ and

$$\begin{aligned} V^I(N_I, N_E) &= \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta} \\ &\quad + \delta(V^I(N_I + 1, N_E) - V^E(N_I, N_E + 1)) \end{aligned}$$

From the induction hypothesis, we have

$$V^I(N_I{+}1, N_E) = \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_I - N_I - 2} (j+1)\delta^j f_I(N_I + 1 + j) & \text{if} \quad s_E - s_I < \Delta^I(N_I + 1, N_E + 1) \\ \frac{s_I - s_E + \Delta^I(N_I + 1, N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I + 1, N_E + 1) \le s_E - s_I < \Delta^I(N_I + 1, N_E) \\ 0 & \text{if} \quad s_E - s_I \ge \Delta^I(N_I + 1, N_E) \end{cases}$$

$$V^E(N_I, N_E{+}1) = \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E + 1) \\ \frac{s_E - s_I - \Delta^I(N_I, N_E + 1)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I, N_E + 1) \le s_E - s_I < \Delta^I(N_I + 1, N_E + 1) \\ \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 2} \delta^j f_E(N_E + 1 + j) - \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} & \text{if} \quad s_E - s_I \ge \Delta^I(N_I + 1, N_E + 1) \end{cases} .$$

Plugging these two expressions into the expression of $V^I(N_I, N_E)$ above, we obtain

$$V^I(N_I, N_E) = \frac{s_I - s_E + (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - f_E(N_E)}{1-\delta}$$

$$+\delta \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_I - N_I - 2} (j+1)\delta^j f_I(N_I + 1 + j) & \text{if} \quad s_E - s_I < \Delta^I(N_I + 1, N_E + 1) \\ \frac{s_I - s_E + \delta^{\overline{N}_I - N_I - 1}(\overline{N}_I - N_I - (\overline{N}_I - N_I - 1)\delta) f_I(\overline{N}_I) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)}{(1-\delta)^3} \\ \quad + \frac{\sum_{j=0}^{\overline{N}_I - N_I - 2} (j+1)\delta^j f_I(N_I + 1 + j)}{1-\delta} \\ \quad - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j)}{(1-\delta)^2} & \text{if} \quad \Delta^I(N_I + 1, N_E + 1) \le s_E - s_I < \Delta^I(N_I + 1, N_E) \\ 0 & \text{if} \quad s_E - s_I \ge \Delta^I(N_I + 1, N_E) \end{cases}$$

$$
-\delta \begin{cases} 0 & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\[2ex] \frac{s_E - s_I - \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E)}{(1-\delta)^3} \\ \quad - \frac{\sum_{j=0}^{\overline{N}_E - N_E - 1}(j+1)\delta^j f_I(N_I + j)}{1-\delta} & \text{if} \quad \begin{array}{c} \Delta^I(N_I, N_E+1) \leq s_E - s_I \\ < \Delta^I(N_I+1, N_E+1) \end{array} \\ \quad + \frac{\sum_{j=0}^{n-2} \delta^j f_E(N_E + 1 + j)}{(1-\delta)^2} \\[2ex] \frac{s_E - s_I + \delta^{\overline{N}_E - N_E - 1} f_E(\overline{N}_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad + \frac{\sum_{j=0}^{\overline{N}_E - N_E - 2} \delta^j f_E(N_E + 1 + j) - \sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} & \text{if} \quad s_E - s_I \geq \Delta^I(N_I+1, N_E+1) \end{cases}
$$

Straightforward calculations and the imposition of the condition $V^I(N_I, N_E) \geq 0$ lead to

$$
V^I(N_I, N_E) = \begin{cases} \frac{s_I - s_E + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) - f_E(N_E)}{(1-\delta)^2} \\ \quad + \sum_{j=0}^{\overline{N}_I - N_I - 1}(j+1)\delta^j f_I(N_I + j) & \text{if} \quad s_E - s_I < \Delta^I(N_I, N_E+1) \\[2ex] \frac{s_I - s_E + \Delta^I(N_I, N_E)}{(1-\delta)^3} & \text{if} \quad \begin{array}{c} \Delta^I(N_I, N_E+1) \leq s_E - s_I \\ < \Delta^I(N_I, N_E) \end{array} \\[2ex] 0 & \text{if} \quad s_E - s_I \geq \Delta^I(N_I, N_E) \end{cases},
$$

where

$$
\begin{aligned}
\Delta^I(N_I, N_E) \quad = \quad & (1-\delta)^2 \sum_{j=0}^{\overline{N}_I - N_I - 1} (j+1)\delta^j f_I(N_I + j) + \delta^{\overline{N}_I - N_I}(\overline{N}_I - N_I + 1 - (\overline{N}_I - N_I)\delta) f_I(\overline{N}_I) \\
& -(1-\delta) \sum_{j=0}^{\overline{N}_E - N_E - 1} \delta^j f_E(N_E + j) - \delta^{\overline{N}_E - N_E} f_E(\overline{N}_E)
\end{aligned}
$$

Now suppose E wins the current period, so $V^I(N_I, N_E) = V^I(N_I, N_E+1) = 0$ and

$$
\begin{aligned}
V^E(N_I, N_E) \quad = \quad & \frac{s_E - s_I + f_E(N_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta} \\
& + \delta(V^E(N_I, N_E+1) - V^I(N_I+1, N_E)).
\end{aligned}
$$

We already have the expressions of $V^I(N_I+1, N_E)$ and $V^E(N_I, N_E+1)$ from the induction hypothesis above. Plugging these two expressions into the expression of $V^E(N_I, N_E)$, we obtain

$$
V^E(N_I, N_E) = \frac{s_E - s_I + f_E(N_E) - \delta^{\overline{N}_I - N_I} f_I(\overline{N}_I) - (1-\delta)\sum_{j=0}^{\overline{N}_I - N_I - 1} \delta^j f_I(N_I + j)}{1-\delta}
$$

$$-\delta\begin{cases} \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-f_E(N_E)}{(1-\delta)^2} \\ \quad +\sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j) & \text{if} \quad s_E-s_I<\Delta^I(N_I+1,N_E+1) \\ \\ \frac{s_I-s_E+\delta^{\overline{N}_I-N_I-1}(\overline{N}_I-N_I-(\overline{N}_I-N_I-1)\delta)f_I(\overline{N}_I)-\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)}{(1-\delta)^3} \\ \quad +\frac{\sum_{j=0}^{\overline{N}_I-N_I-2}(j+1)\delta^j f_I(N_I+1+j)}{1-\delta} & \text{if} \quad \Delta^I(N_I+1,N_E+1)\leq s_E-s_I \\ \quad -\frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)}{(1-\delta)^2} & \quad <\Delta^I(N_I+1,N_E) \\ \\ 0 & \text{if} \quad s_E-s_I\geq\Delta^I(N_I+1,N_E) \end{cases}$$

$$+\delta\begin{cases} 0 & \text{if} \quad s_E-s_I<\Delta^I(N_I,N_E+1) \\ \\ \frac{s_E-s_I-\delta^{\overline{N}_I-N_I}(\overline{N}_I-N_I+1-(\overline{N}_I-N_I)\delta)f_I(\overline{N}_I)+\delta^{\overline{N}_E-N_E-1}f_E(\overline{N}_E)}{(1-\delta)^3} \\ \quad -\frac{\sum_{j=0}^{\overline{N}_I-N_I-1}(j+1)\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad \Delta^I(N_I,N_E+1)\leq s_E-s_I \\ \quad +\frac{\sum_{j=0}^{\overline{N}_E-N_E-2}\delta^j f_E(N_E+1+j)}{(1-\delta)^2} & \quad <\Delta^I(N_I+1,N_E+1) \\ \\ \frac{s_E-s_I+\delta^{\overline{N}_E-N_E-1}f_E(\overline{N}_E)-\delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad +\frac{\sum_{j=0}^{\overline{N}_E-N_E-2}\delta^j f_E(N_E+1+j)-(1-\delta)\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad s_E-s_I\geq\Delta^I(N_I+1,N_E+1) \end{cases}$$

Straightforward calculations and the imposition of the condition $V^E(N_I,N_E)\geq 0$ lead to

$$V^E(N_I,N_E)=\begin{cases} 0 & \text{if} \quad s_E-s_I<\Delta^I(N_I,N_E) \\ \\ \frac{s_E-s_I-\Delta(N_I,N_E)}{(1-\delta)^3} & \text{if} \quad \Delta^I(N_I,N_E)\leq s_E-s_I \\ & \quad <\Delta^I(N_I+1,N_E) \\ \\ \frac{s_E-s_I+\delta^{\overline{N}_E-N_E}f_E(\overline{N}_E)-\delta^{\overline{N}_I-N_I}f_I(\overline{N}_I)}{(1-\delta)^2} \\ \quad +\frac{\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)-\sum_{j=0}^{\overline{N}_I-N_I-1}\delta^j f_I(N_I+j)}{1-\delta} & \text{if} \quad s_E-s_I\geq\Delta^I(N_I+1,N_E) \end{cases}.$$

Thus, the result holds for the state (N_I,N_E). Using this induction reasoning repeatedly, we conclude that the result holds for every state (N_I,N_E) with $0\leq N_I\leq\overline{N}_I$ and $0\leq N_E\leq\overline{N}_E$.

The same reasoning as in the case with Pareto beliefs ensure that consumers do not also buy from the losing firm.

Finally, we wish to prove that $\Delta^I(N_I,N_E)>\Delta(N_I,N_E)$. This is equivalent to

$$(1-\delta)^2\sum_{j=0}^{\overline{N}_E-N_E-1}(j+1)\delta^j f_E(N_E+j)+(\overline{N}_E-N_E)\delta^{\overline{N}_E-N_E}(1-\delta)f_E(\overline{N}_E)-(1-\delta)\sum_{j=0}^{\overline{N}_E-N_E-1}\delta^j f_E(N_E+j)>0.$$

Dividing this through by $(1-\delta)$ and cancelling out some terms, we can rewrite the inequality as

$$\sum_{j=1}^{\overline{N}_E - N_E} \left(j\delta^j (f_E(N_E + j) - f_E(N_E + j - 1))\right) > 0$$

which is true given $f(.)$ is increasing (indeed, strictly increasing for at least one step). Thus, we have proven that $\Delta^I(N_I, N_E) > \Delta(N_I, N_E)$.

G Detailed analysis from Section 5.3

Here, we show that the only possible pure-strategy equilibria are (i) all consumers choose E in every period, and (ii) all consumers choose I in every period. Suppose there is a pure-strategy equilibrium in which consumers choose I in some periods and E in other periods. In any such equilibrium consumers must switch from one firm to the other. Given the firms are symmetric, suppose consumers choose I for the first m_I periods and switch to E in period $m_I + 1$. To keep the notation streamlined, define

$$\overline{f}^W(n) \equiv \begin{cases} f^W(n) & \text{if} \quad n \leq \overline{n} \\ f^W(\overline{n}) & \text{if} \quad n > \overline{n} \end{cases}.$$

Suppose $m_I = 1$, i.e. the equilibrium involves all consumers buying from I in period 1 and E in period 2. But then, if $s_E \geq s_I$, a consumer would do better to deviate from the proposed equilibrium and choose E in all periods, while if $s_I > s_E$, a consumer would do better to deviate from the proposed equilibrium and choose I in all periods.

Thus, we must have $m_I \geq 2$. In all periods starting with period $m_I + 2$, consumers get the same across-user learning value $f^A(1)$ on both firms because all consumers have chosen each firm for at least one period. This means that in all periods starting with period $m_I + 2$, each consumer chooses I if

$$\frac{s_I}{1-\delta} + \sum_{j=m_I}^{\infty} \delta^{j-m_I} \overline{f}^W(j) > \frac{s_E}{1-\delta} + \sum_{j=1}^{\infty} \delta^{j-1} \overline{f}^W(j), \tag{G.1}$$

and E otherwise. In other words, consumers switch at most twice in equilibrium.

Suppose the inequality above holds, so consumers switch back to I in period $m_I + 2$, after one period of choosing E. But then it is easily seen that a consumer would do better to deviate from the proposed equilibrium path and choose I in all periods. Indeed, by doing so, the consumer's PDV from the perspective of period $m_I + 1$ is

$$\frac{s_I}{1-\delta} + \sum_{j=m_I}^{\infty} \delta^{j-m_I} \overline{f}^W(j),$$

whereas the equilibrium PDV is

$$s_E + \delta\left(\frac{s_I}{1-\delta} + \sum_{j=m_I}^{\infty} \delta^{j-m_I}\overline{f}^W(j)\right).$$

It is easily verified that the former is greater than the latter when inequality (G.1) holds.

Thus, it must be that

$$\frac{s_I}{1-\delta} + \sum_{j=m_I}^{\infty} \delta^{j-m_I}\overline{f}^W(j) \leq \frac{s_E}{1-\delta} + \sum_{j=1}^{\infty} \delta^{j-1}\overline{f}^W(j),$$

so the proposed equilibrium has all consumers choose I for the first m_I periods and E in all periods thereafter.

A consumer's payoff on this proposed equilibrium's path is then

$$s_I\frac{1-\delta^{m_I}}{1-\delta} + s_E\frac{\delta^{m_I}}{1-\delta} + \left(\frac{\delta}{1-\delta} - \delta^{m_I}\right) f^A(1) + \sum_{j=1}^{m_I-1} \delta^j\overline{f}^W(j) + \delta^{m_I}\sum_{j=1}^{\infty} \delta^j\overline{f}^W(j).$$

Consider the family of deviations in which the deviant goes to I for the first $k \geq 1$ periods and then switches to E forever. Note that $k = m_I$ corresponds to sticking to the equilibrium strategy. If $k \leq m_I$, then the deviation payoff is

$$s_I\frac{1-\delta^k}{1-\delta} + s_E\frac{\delta^k}{1-\delta} + \sum_{j=1}^{k-1} \delta^j\overline{f}^W(j) + \delta^k\sum_{j=1}^{\infty} \delta^j\overline{f}^W(j) + f^A(1)\frac{\delta\left(1-\delta^{k-1}\right) + \delta^{m_I+1}}{1-\delta}.$$

If $k \geq m_I + 1$, then the deviation payoff is

$$s_I\frac{1-\delta^k}{1-\delta} + s_E\frac{\delta^k}{1-\delta} + \sum_{j=1}^{k-1} \delta^j\overline{f}^W(j) + \delta^k\sum_{j=1}^{\infty} \delta^j\overline{f}^W(j) + f^A(1)\frac{\delta}{1-\delta}.$$

The difference between the equilibrium payoff ($k = m_I$) and the payoff with $k = m_I + 1$ is

$$(s_E - s_I)\delta^{m_I} - \delta^{m_I}\overline{f}^W(m_I) + \delta^{m_I}(1-\delta)\sum_{j=1}^{\infty} \delta^j\overline{f}^W(j) - f^A(1)\delta^{m_I}.$$

Thus, for the deviation to $k = m_I + 1$ to not be profitable, we need

$$s_E - s_I \geq f^W(m_I) - (1-\delta)\sum_{j=1}^{\infty} \delta^j f^W(j) + f^A(1).$$

Similarly, the difference between the equilibrium payoff ($k = m_I$) and the payoff with $k = m_I - 1$

is

$$(s_I - s_E)\,\delta^{m_I-1} + \delta^{m_I-1}\overline{f}^W(m_I-1) - (1-\delta)\,\delta^{m_I-1}\sum_{j=1}^{\infty}\delta^j\overline{f}^W(j) + f^A(1)\,\delta^{m_I-1}.$$

Thus, for the deviation to $k = m_I - 1$ to not be profitable, we need

$$s_E - s_I \leq \overline{f}^W(m_I-1) - (1-\delta)\sum_{j=1}^{\infty}\delta^j\overline{f}^W(j) + f^A(1)\,.$$

Since $\overline{f}^W(.)$ is weakly increasing, the only way that the last two inequalities can be satisfied is if $\overline{f}^W(m_I-1) = \overline{f}^W(m_I)$ and $s_E - s_I$ is such that the payoffs to choosing $k = m_I - 1$, $k = m_I$ and $k = m_I + 1$ are the same, i.e.

$$s_E - s_I = \overline{f}^W(m_I) - (1-\delta)\sum_{j=1}^{\infty}\delta^j\overline{f}^W(j) + f^A(1)\,.$$

But then we can repeat the analysis starting from $k = m_I - 1$ (which yields the same payoff as the equilibrium strategy), and comparing it with $k = m_I - 2$ and $k = m_I$. In order for $k = m_I - 2$ not to be a profitable deviation, we then need $\overline{f}^W(m_I-2) = \overline{f}^W(m_I-1) = \overline{f}^W(m_I)$. And we can repeat this process of comparisons until either we get to a point where $\overline{f}^W$ is strictly increasing, which then ensures there is a profitable deviation, or we must have $\overline{f}^W(0) = \overline{f}^W(1) = ... = \overline{f}^W(m_I) = 0$.

In the latter case, recall that we must also have

$$\frac{s_I}{1-\delta} + \sum_{j=m_I}^{\infty}\delta^{j-m_I}\overline{f}^W(j) \leq \frac{s_E}{1-\delta} + \sum_{j=1}^{\infty}\delta^{j-1}\overline{f}^W(j)\,,$$

which implies $s_E > s_I$. This means that a consumer would do better to deviate by choosing E in the first period and then following the equilibrium path. Since $\overline{f}^W(0) = \overline{f}^W(1) = ... = \overline{f}^W(m_I) = 0$, there is no loss in terms of within-user learning.

Thus, we have proven that consumers choosing I for the first m_I periods and choosing E in period $m_I + 1$ cannot be an equilibrium for any $m_I \geq 1$. By symmetry, there can be no equilibrium in which consumers choose E for the first $m_E \geq 1$ periods and choose I in period $m_E + 1$. We conclude therefore that the only possible equilibria are either consumers choose I in all periods or choose E in all periods.