# Online Appendix for "Signaling Private Choices"

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This supplementary online appendix provides additional results and formal proofs for the main paper "Signaling Private Choices". Section 1 provides a formal proof of Proposition 1 and Corollary 1 in Section 4 of the main paper. Section 2 provides the full details for each of the applications in Section 4.2 of the main paper. Section 3 provides a technical appendix to Section 5 of the main paper. It provides some technical preliminaries for a more formal treatment of endogenous signaling games, clarifies some technical concepts in the definition of the class of the games, provides the reordering algorithm, provides a guide to reordering for applied researchers, proves Proposition 4 of the main paper, and provides an example to show the relationship with proper equilibrium.

## 1 Proof of Proposition 1 and Corollary 1

In order to prove Proposition 1 of the main paper, we introduce some notation and establish a lemma. For games in $\Gamma_S^1$, we assume without loss of generality that $T$, $A$, and $B$ are subsets of the set of real numbers. We use Greek letters $\tau$, $\alpha$, and $\beta$ for probability distributions on $T$, $A$, and $B$ respectively ($\tau \in \Delta T$, $\alpha \in \Delta A$, and $\beta \in \Delta B$). For any game in $\Gamma_S^1$, the sender's behavior strategy[1] is in the form of $(\tau, \alpha(\cdot))$ and the receiver's behavior strategy is in the form of $\beta(\cdot)$. We extend the payoff functions $\tilde{u}_S$ and $\tilde{u}_R$ to behavior strategies such that[2]

$$\begin{aligned}
\tilde{u}_S(\tau, \alpha(\cdot), \beta(\cdot)) &\equiv \int_T \int_A \int_B u_S(t,a,b)\,\beta(a)(db)\,\alpha(t)(da)\,\tau(dt), \\
\tilde{u}_R(\tau, \alpha(\cdot), \beta(\cdot)) &\equiv \int_T \int_A \int_B u_R(t,a,b)\,\beta(a)(db)\,\alpha(t)(da)\,\tau(dt).
\end{aligned}$$

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[1]Aumann (1964) extended Kuhn's (1953) theorem (on the equivalence of mixed strategies and behavior strategies) to infinite extensive-form games.

[2]Note that payoff functions $\tilde{u}_S$ and $\tilde{u}_R$ are functions of pure-strategy profiles whereas payoff functions $u_S$ and $u_R$ are functions of terminal histories. See Section 3.1.

Similarly, let the expected payoffs in the subgame of the reordered game which follows the choice of $a$ (when the sender chooses $\tau(a)$ and the receiver chooses $\beta(a)$) be denoted by

$$\begin{aligned} u_S(\tau(a), a, \beta(a)) &\equiv \int_T \int_B u_S(t, a, b)\, \beta(a)(db)\, \tau(a)(dt), \\ u_R(\tau(a), a, \beta(a)) &\equiv \int_T \int_B u_R(t, a, b)\, \beta(a)(db)\, \tau(a)(dt). \end{aligned}$$

Suppose that $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is a subgame-perfect equilibrium in $G_S^1 \in \Gamma_S^1$. From $\tau^o$ and $\alpha^o(\cdot)$, we can construct a joint probability distribution on $T \times A$. Let $\alpha^o$ be the marginal probability distribution (of $a$) induced by the joint probability distribution. Let $A^o$ be the support of $\alpha^o$, and a typical subset of $A^o$ which has $\alpha^o$-measure zero be denoted by $A^o_{null}$. Let $\tau^o(a)$ be the conditional probability distribution (of $t$ given $a \in A^o$), induced by the joint probability distribution.

**Lemma 1** *Consider any game $G_S^1 \in \Gamma_S^1$ and its reordered game $G_r^1$. If a strategy profile $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is a subgame-perfect equilibrium in $G_S^1$ then for all $a^o \in A^o \setminus A^o_{null}$ for some $A^o_{null} \subset A^o$, $(\tau^o(a^o), \beta^o(a^o))$ is a Nash equilibrium in the subgame of $G_r^1$ following $a^o$.*

**Proof of Lemma 1.** Since $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is a subgame-perfect equilibrium in $G_S^1$,

$$\begin{aligned} (\tau^o, \alpha^o(\cdot)) &\in \arg \max_{(\tau, \alpha(\cdot)) \in \Delta T \times (\Delta A)^T} \tilde{u}_S(\tau, \alpha(\cdot), \beta^o(\cdot)) \quad \text{and} \\ \beta^o(\cdot) &\in \arg \max_{\beta(\cdot) \in (\Delta B)^A} \tilde{u}_R(\tau^o, \alpha^o(\cdot), \beta(\cdot)). \end{aligned}$$

Then we can choose $A^o_{null} \subset A^o$ such that for all $a^o \in A^o \setminus A^o_{null}$,

$$\begin{aligned} \tau^o(a^o) &\in \arg \max_{\tau \in \Delta T} u_S(\tau, a^o, \beta^o(a^o)), & (1) \\ \beta^o(a^o) &\in \arg \max_{\beta \in \Delta B} u_R(\tau^o(a^o), a^o, \beta), \quad \text{and} & (2) \end{aligned}$$

$$u_S(\tau^o(a^o), a^o, \beta^o(a^o)) = \tilde{u}_S(\tau^o, \alpha^o(\cdot), \beta^o(\cdot)). \tag{3}$$

The inclusions (1) and (2) imply that $(\tau^o(a^o), \beta^o(a^o))$ is a Nash equilibrium in the subgame of $G_r^1$ following $a^o$. ■

**Proof of Proposition 1.** We first prove a claim that for any subgame-perfect equilibrium $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ in $G_S^1$, there exists an *RI-equilibrium* where the sender's payoff is greater than or equal to the sender's payoff in this subgame-perfect equilibrium. Consider an *RI-equilibrium* $(\tilde{\tau}, \tilde{\alpha}(\cdot); \tilde{\beta}(\cdot))$ in $G_S^1$. This implies that there exists a subgame-perfect equilibrium $(\alpha^*, \tau^*(\cdot); \beta^*(\cdot))$ in $G_r^1$, where $\beta^*(\cdot) = \tilde{\beta}(\cdot)$. If the sender's payoff in this *RI-equilibrium* is greater than or equal to its payoff in $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ then we are done. Suppose not. Consider a strategy profile $(a^o, \hat{\tau}(\cdot); \hat{\beta}(\cdot))$ in $G_r^1$ such that

$$\hat{\tau}(a) = \begin{cases} \tau^o(a^o) & \text{for } a = a^o \\ \tau^*(a) & \text{for } a \neq a^o \end{cases} \quad \text{and} \quad \hat{\beta}(a) = \begin{cases} \beta^o(a^o) & \text{for } a = a^o \\ \beta^*(a) & \text{for } a \neq a^o, \end{cases}$$

where $a^o \in A^o$ is chosen such that the inclusions (1), (2), and the equation (3) hold. This strategy profile is a subgame-perfect equilibrium in $G_r^1$ because (i) $(\tau^o(a^o), \beta^o(a^o))$ is a Nash equilibrium in the subgame following $a^o$ by Lemma 1, (ii) $(\tau^*(a), \beta^*(a))$ is a Nash equilibrium in the subgame following $a \neq a^o$, and (iii) $a^o$ is an optimal choice for the sender given the Nash equilibria in the proper subgames (since the sender's payoff in the *RI-equilibrium* is less than its payoff in the subgame-perfect equilibrium $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$). We can verify that $(\hat{\tau}(a^o), a^o; \hat{\beta}(\cdot))$ is an *RI-equilibrium* in $G_S^1$, where the choice of $a$ is always $a^o$ regardless of the choice of $t$. This *RI-equilibrium* yields the same payoff to the sender as $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$. This proves the claim. Since the set of *RI-equilibrium* payoffs to the sender admits a maximal element, there exists at least one *RI-equilibrium* that yields the best payoff to the sender among all subgame-perfect equilibria.

Now it remains to prove the last statement. Suppose that a subgame-perfect equilibrium $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ yields a payoff to the sender which is the same as or greater than that of an *RI-equilibrium* in $G_S^1$ . Corresponding to this *RI-equilibrium*, there exists a subgame-perfect equilibrium $(\alpha^*, \tau^*(\cdot); \beta^*(\cdot))$ in $G_r^1$, where $\beta^*(\cdot)$ is the same as the receiver's strategy in the *RI-equilibrium* and the sender's payoff is not greater than that of the *RI-equilibrium.* We prove the last statement by showing that the subgame-perfect equilibrium $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is outcome-equivalent to an *RI-equilibrium* in $G_S^1$. We can choose $A^o_{null} \subset A^o$ and construct a strategy profile $(\alpha^o, \hat{\tau}(\cdot); \hat{\beta}(\cdot))$ in $G_r^1$ such that

$$\hat{\tau}(a) = \begin{cases} \tau^o(a) & \text{for } a \in A^o \setminus A^o_{null} \\ \tau^*(a) & \text{for } a \notin A^o \setminus A^o_{null} \end{cases} \quad \text{and} \quad \hat{\beta}(a) = \begin{cases} \beta^o(a) & \text{for } a \in A^o \setminus A^o_{null} \\ \beta^*(a) & \text{for } a \notin A^o \setminus A^o_{null}. \end{cases}$$

This strategy profile is a subgame-perfect equilibrium in $G_r^1$ because (i) $(\tau^o(a), \beta^o(a))$ is a Nash equilibrium in the subgame following $a \in A^o \setminus A^o_{null}$ by Lemma 1, (ii) $(\tau^*(a), \beta^*(a))$ is a Nash equilibrium in the subgame following $a \notin A^o \setminus A^o_{null}$, and (iii) $\alpha^o$ is an optimal choice for the sender given the Nash equilibria in the proper subgames (since $\max_a\{u_S(\tau^*(a), a, \beta^*(a))\}$ is not greater than $\max_a\{u_S(\tau^o(a), a, \beta^o(a))\}$). The subgame-perfect equilibrium $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is outcome-equivalent[3] to the subgame-perfect equilibrium $(\alpha^o, \hat{\tau}(\cdot); \hat{\beta}(\cdot))$ in $G_r^1$, and outcome-equivalent to an *RI-equilibrium* $(\tau^o, \alpha^o(\cdot); \hat{\beta}(\cdot))$ in $G_S^1$. ■

**Proof of Corollary 1.** Note that the best payoff to the sender among all Nash equilibrium payoffs in the proper subgames of $G_r^1$ is the best payoff to the sender among all subgame-perfect equilibrium payoffs in $G_S^1$. Suppose on the contrary that there exists an *RI-equilibrium*

[3]We say two strategy profiles are *outcome-equivalent* if probability distributions over the set of terminal nodes are equal almost everywhere when the one-to-one correspondence of terminal nodes between the original and reordered games is considered.

$(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ that yields a strictly lower payoff to the sender than its best equilibrium payoff. By the definition of *RI-equilibrium*, $\beta^o(a)$ is a Nash equilibrium strategy in the subgame following each $a$ in $G^1_r$. Suppose that the subgame following $a^*$ in $G^1_r$ is the proper subgame where all Nash equilibria yield the same best payoff to the sender. Let $(\tau^*(a^*); \beta^o(a^*))$ be one of the Nash equilibria in the proper subgame. This implies that the sender can obtain its best equilibrium payoff by choosing some $t$ in the support of $\tau^*(a^*)$ and then $a^*$ when the receiver chooses its strategy $\beta^o(\cdot)$ in $G^1_S$. Let $t^*$ be one such $t$. Then, the sender can deviate profitably from its strategy $(\tau^o, \alpha^o(\cdot))$ by choosing $t^*$ at its first information set and $a^*$ at all of its succeeding information sets in $G^1_S$. This is a contradiction to the supposition that the strategy profile $(\tau^o, \alpha^o(\cdot); \beta^o(\cdot))$ is an *RI-equilibrium.* ■

# 2 Applications

In this section, we provide the full details of the applications discussed in Section 4.2 of the main paper. As will become clear from the models that follow, the properties of monotone endogenous signaling games do not hold globally for these applications. Nevertheless, since they hold locally around the relevant equilibria for appropriate parameter values, the commitment effect and signal exaggeration still apply for appropriate parameter values.

## 2.1 Costly announcements and inflation

Consider the game specified in Section 4.2.1 of the main paper.

In the original game there are a continuum of subgame-perfect equilibria, which are difficult to characterize in general. Focusing on pure-strategy equilibria, and assuming differentiability of the receiver's equilibrium strategy $\bar{b}(\cdot)$ and second-order conditions hold, the equilibria $(\bar{t}, \bar{a}(\cdot); \bar{b}(\cdot))$ have to satisfy

$$\begin{aligned} \bar{t} &= \frac{\kappa \bar{a}(\bar{t}) + \lambda \hat{t} - \theta}{\kappa + \lambda}, &(4)\\ \bar{a}(\bar{t}) &= \bar{t} - \frac{\left(\bar{b}(\bar{a}) - (\bar{t} + \theta)\right) \bar{b}'(\bar{a})}{\kappa}, \text{ and} &(5)\\ \bar{b}(\bar{a}) &= \bar{t}, &(6) \end{aligned}$$

where recall $\bar{a} = \bar{a}(\bar{t})$ and $\bar{b} = \bar{b}(\bar{a})$. Even assuming linearity of the receiver's equilibrium strategy, *i.e.* $\bar{b}(a) = c_0 + c_1 a$, we obtain a continuum of equilibria satisfying (4)-(6).[4] These

[4] Going beyond linearity (or differentiability) of the receiver's beliefs $t^e(\cdot)$, additional equilibria may arise and second-order conditions will no longer be straightforward to confirm.

equilibria can be parameterized by $c_1$. The corresponding equilibrium outcomes are:

$$\bar{t} = \bar{b} = \hat{t} - \frac{\theta(1-c_1)}{\lambda}, \tag{7}$$

$$\bar{a} = \hat{t} - \frac{\theta}{\lambda} + \frac{\theta(\lambda+\kappa)c_1}{\kappa\lambda}, \text{ and} \tag{8}$$

$$u_S(\bar{t},\bar{a},\bar{b}) = -\frac{\theta^2\left(\kappa(1-c_1)^2 + \lambda\left(\kappa + c_1^2\right)\right)}{\kappa\lambda}. \tag{9}$$

By setting $c_1 = 0$, we obtain the equilibrium outcome with passive beliefs $t^{pa} = a^{pa} = b^{pa} = \hat{t} - \theta/\lambda$. Since the sender's announcement is ignored, it may as well tell the truth. However, in equilibrium it chooses a lower level of effort compared to its ideal level $t^{ob} = \hat{t}$ due to its lack of commitment (if it could commit to $t = \hat{t}$ it would be better off). This corresponds to the discretionary inflation level (higher than first-best) in the time inconsistency example.

Solving the reordered game, in which $a$ is set in stage 1a and $t$ is set in stage 1b, is trivial. We first look for an equilibrium in the choice of $t$ and $b$ for a given choice of $a$. The result is that $t^*(a) = b^*(a) = (\kappa a - \theta + \hat{t}\lambda)/(\lambda+\kappa)$ for all $a$. Given this, solving for the optimal choice of $a$ in stage 1a, the unique *RI-equilibrium* outcome of the original game is $\tilde{t} = \tilde{b} = \hat{t} - \theta/(\lambda+\kappa)$ and $\tilde{a} = \hat{t}$. Substituting $c_1 = \kappa/(\lambda+\kappa)$ into (7)-(8) confirms this outcome corresponds to one of the continuum of equilibria above, parameterized by $c_1$.

Consistent with optimality, the *RI-equilibrium* involves the best payoff for the sender from the continuum of equilibrium payoffs in (9). Clearly, if $\theta > 0$ then $\tilde{t} > t^{pa}$ and $\tilde{a} > a^{pa}$ so that even though the sender's effort is unobserved, it is still able to obtain a commitment benefit in the *RI-equilibrium*. This corresponds to reducing inflation towards the government's first-best level in the time inconsistency example, *i.e.* $\tilde{t} < t^{pa}$ and $\tilde{a} < a^{pa}$ reflecting that $\theta < 0$ in the example. Note, in the *RI-equilibrium* the sender benefits from having a higher lying cost parameter $\kappa$.

Consistent with signal exaggeration, we also find announcements are inflated: $\tilde{a} = \hat{t} > a^{ob}(\tilde{t}) = \tilde{t} = \hat{t} - \theta/(\lambda+\kappa)$ if $\theta > 0$. Although the sender still chooses less effort than its first-best level $\hat{t}$, due to signal exaggeration it turns out that its announced effort level $a$ is exactly equal to its ideal level $\hat{t}$. As a consequence, in this example, while signal exaggeration increases as the lying cost parameter $\kappa$ gets small, the level of signal exaggeration is bounded—the equilibrium level of effort converges towards the passive beliefs level $\hat{t} - \theta/\lambda$, while the announced effort remains at $\hat{t}$. Even with a very small cost of lying, the sender will not want to exaggerate very much.

## 2.2 Limit pricing and business strategy

A large set of applications of endogenous signaling games is to the case of unobserved "investment". In this section, we provide the full details of the model of limit pricing briefly noted in Section 4.2.2 of the main paper, in which the incumbent's private cost is determined endogenously by its unobserved investment in cost-reducing R&D. This application, along with other applications involving the signaling of private investment choices, are a cross between the top-dog entry deterrence strategy of Fudenberg and Tirole (1984) and the classic model of limit pricing studied by Milgrom and Roberts (1982), as shown in Table 1.

<table>
<tr><th></th><th>Observed</th><th>Unobserved</th></tr>
<tr><td>Nature chooses</td><td></td><td>Limit pricing<br>Milgrom & Roberts (1982)<br>Signaling in oligopoly<br>Mailath (1989)</td></tr>
<tr><td>Sender chooses</td><td>Business strategy<br>Fudenberg & Tirole (1984)</td><td>Signaling private investment choices</td></tr>
</table>

Table 1: Business strategy and signaling

The specific model of limit pricing we consider involves an incumbent $I$ that initially has constant marginal cost $c_H$ and is a monopolist. It faces a potential rival $E$ with marginal cost $c_L < c_H$. The timing, information and payoffs are summarized as follows:

1. In stage 1, $I$ chooses how much to invest $k \in [0, K]$ in some potentially cost-reducing technology, where $K > 0$. The cost of this investment is $C(k)$, which is increasing and convex. Specifically, we assume $C(0) = 0$, $C'(0) \geq 0$, $C''(0) \geq 0$, $C'(k) > 0$ and $C''(k) > 0$ for $0 < k \leq K$. The investment has probability $\alpha(k)$ of being successful, where $\alpha'(k) > 0$ and $\alpha''(k) < 0$ for $0 \leq k < K$, with $\alpha(0) = 0$ and $\alpha(K) = 1$. With probability $0 \leq \alpha(k) \leq 1$, $I$'s constant marginal cost will be $c_L < c_H$, while with probability $1 - \alpha(k)$, $I$'s marginal costs will remain at $c_H$. Thus, the more the incumbent invests, the higher the probability of it being able to lower its costs from $c_H$ to $c_L$, with this probability becoming one if it invests $K$. Without observing the outcome of whether its investment is successful, $I$ also sets a price $p > 0$, reflecting it has to announce a price and cannot change it over the relevant period prior to a rival's possible entry.

2. In stage 2, the success of the investment is realized, and $I$ produces for the relevant demand $Q(p)$. We assume $Q(p)$ is continuous in $p$ for positive $p$, and strictly decreasing in $p$ until demand becomes zero.

3. In stage 3, a potential entrant $E$ observes $p$ (but not $k$ or the outcome of the investment) and decides whether to enter. It faces a constant marginal cost of $c_L$, but it also has to incur a fixed cost $F > 0$ of entry. Its outside option is zero.

4. In stage 4, $E$ also learns whether $I$'s investment is successful or not. If $E$ has entered, the two firms compete in prices according to standard (possibly asymmetric) Bertrand price competition. Otherwise, $I$ remains a monopolist. The market demand function $Q$

is the same as before but is multiplied by $\beta$, which if greater than one reflects that the final period actually represents many periods.

Let $p^m(c)$ denote the standard monopoly price which maximizes $(p-c)\,Q(p)$, which we assume is uniquely defined for $c_L \leq c \leq c_H$, and assume $p^m(c_H) > p^m(\alpha c_L + (1-\alpha)\,c_H) > p^m(c_L) > c_H$ for $0 < \alpha < 1$. The assumption just states the standard property that price is strictly increasing in marginal cost, and that $I$'s innovation is not drastic. The corresponding one-period monopoly profit is denoted $\pi^m(c) = (p^m(c) - c)\,Q(p^m(c))$. Let $\Delta\pi^m = \pi^m(c_L) - \pi^m(c_H) > 0$ be the gain in monopoly profit from successful innovation. Note that if it expects an investment of $k$, $E$'s expected profit is

$$\Pi_E = (1-\alpha(k))\,(c_H - c_L)\,Q(c_H) - F$$

since due to Bertrand competition it only makes a profit if it has a cost advantage. Assume $(c_H - c_L)\,Q(c_H) > F$ so that $E$ would want to enter if it knew $I$'s costs would remain at $c_H$. This ensures there exists $0 < k^e < K$ such that $\Pi_E(k^e) = 0$, so that $E$ enters if and only if it expects $k < k^e$.

To solve for the equilibrium of this game, we consider the reordered game in which $I$ sets its price $p$ first, before it sets $k$.

If $I$ expects entry, its expected profit is

$$\Pi_I^c(p,k) = (p - (\alpha(k)\,c_L + (1-\alpha(k))\,c_H))\,Q(p) - C(k). \qquad (10)$$

Note because of Bertrand competition after entry, $I$ does not obtain any profit after entry. Differentiating $\Pi_I^c$ with respect to $k$ for a given $p$ implies

$$\frac{d\Pi_I^c}{dk} = \alpha'(k)\,(c_H - c_L)\,Q(p) - C'(k).$$

We assume there exists a unique $\underline{p}$ such that

$$\alpha'(k^e)\,(c_H - c_L)\,Q(\underline{p}) = C'(k^e).$$

If $I$ does not expect entry, its expected profit is

$$\begin{aligned}\Pi_I^m(p,k) &= (p - (\alpha(k)\,c_L + (1-\alpha(k))\,c_H))\,Q(p) - C(k) \\ &\quad + \beta\,(\alpha(k)\,\pi^m(c_L) + (1-\alpha(k))\,\pi^m(c_H)).\end{aligned} \qquad (11)$$

Comparing (11) with (10), clearly $I$ does better without entry for any given $k$ and $p$. Differentiating $\Pi_I^m$ with respect to $k$ for a given $p$ implies

$$\frac{d\Pi_I^m}{dk} = \alpha'(k)\,((c_H - c_L)\,Q(p) + \beta\Delta\pi^m) - C'(k).$$

We assume there exists a unique $\overline{p} < c_H$ such that

$$\alpha' (k^e) ((c_H - c_L) Q (\overline{p}) + \beta \Delta \pi^m) = C' (k^e) .$$

This assumption guarantees that for any non-strategic choice of its first-stage price

$$p^m (\alpha (k) c_L + (1 - \alpha (k)) c_H) ,$$

$I$ will not want to invest enough (regardless of its expectation about $E$'s entry decision) to make $E$ want to stay out. This is because $p^m (\alpha (k) c_L + (1 - \alpha (k)) c_H) > c_H > \overline{p} > \underline{p}$, where the inequalities follow from our earlier assumptions.

Given our specification, $I$ has a greater incentive to invest in cost-reducing R&D if it expects to remain a monopolist next period reflecting that it obtains profit in the subsequent period if there is no entry. We can now specify the equilibria in the subgame for any price $p$. There are three regions. In case $p \leq \underline{p}$, then there is a unique equilibrium in which $E$ stays out and $I$ invests some $k \geq k^e$. In case $p > \overline{p}$, then there is a unique equilibrium in which $E$ enters and $I$ invests some $k < k^e$. For prices $\underline{p} < p \leq \overline{p}$, both of these pure strategy equilibria exist. However, selection of the equilibrium without entry is implied by a forward induction argument. If it expects entry, $I$ would do best setting its non-strategic price (a price between $p^m (c_L)$ and $p^m (c_H)$), corresponding to its monopoly price given some $k < k^e$. It could only do better than this by setting $p \leq \overline{p}$ if it expected as a result that $E$ would not enter. Therefore, the fact that it chooses $p \leq \overline{p}$ signals to $E$ that it is coordinating on the equilibrium in which $E$ does not enter, in which it sets $k = k^e$. If $E$ reasons in this way, $E$ would indeed rather not enter.

Finally, consider $I$'s first stage choice of price. It does best either setting its non-strategic price, so there will be entry, or the limit price $\overline{p}$, so as to deter entry. It will choose the latter whenever $\Pi_I^m (\overline{p}, k^e) \geq \max_{p,k} \Pi_I^c (p, k)$.

The equilibrium with entry deterrence involves $I$ overinvesting in the cost-reducing technology compared to the passive-beliefs equilibrium in which $I$'s choice of first-stage price does not affect $E$'s expectation about $k$. This is despite the fact its investment is not observed by $E$. This captures the commitment effect discussed in Section 4.1.1 of the main paper. Indeed, $I$ will invest the same amount as if its investment is fully observable (i.e. $k^e$), although the set of parameter values for which it wants to deter entry is smaller than when its investment is observed, due to the lower profitability of having to engage in limit pricing. Moreover, entry deterrence will involve limit pricing compared to the full information case. Under our assumptions, the incumbent sets its price below its initial cost $c_H$, so as to create an incentive for itself to set a sufficiently high level of $k$ so as to deter entry. This represents signal exaggeration as explained in Section 4.1.2 of the main paper. Note, in case the technology is not successful, this implies the incumbent prices below its marginal cost. Depending on parameter values it could also be that $\overline{p} < c_L$, so pricing is below cost even when the investment in the technology is successful. With its entry successfully deterred, $I$ will raise its price above the limit price in the final stage (i.e. either to $p^m(c_L)$ or $p^m(c_H)$, both of which exceed $c_H$, and so exceed $\overline{p}$). This equilibrium is therefore consistent with predatory pricing.

## 2.3 Quality choice, burning money and advertising

There is a well-established literature looking at the fundamental problem of how a firm convinces consumers of its unobserved quality. In contrast to the exogenous quality literature, to the best of our knowledge, the endogenous quality literature (discussed in the literature review of the main paper) has not considered firms setting both price and advertising as signals of their unobserved quality. This may have been due to a lack of a consistent way to handle consumer beliefs in the face of off-equilibrium prices and advertising levels. Using *Reordering Invariance*, handling multi-dimensional signals is straightforward. In order to compare results with the classical signaling setting as cleanly as possible, we will introduce the simplest possible framework in which to imbed both settings.

There are four stages. In the exogenous quality version of the model, in stage 1 nature determines the quality of the firm (a monopolist) according to $t \in \{L, H\}$, with the probability that $t = H$ given by $0 < \rho < 1$. In the endogenous quality version of the model, in stage 1 the firm (rather than nature) chooses its quality from the same set. The rest of the game remains the same regardless of how quality is chosen. In stage 2, the firm chooses a price $P \geq 0$ and an advertising expenditure $A \in [0, \overline{A}]$ for some sufficiently large $\overline{A}$. In stage 3, a representative consumer (or a continuum of identical consumers) observes these choices but not the quality level and decides whether to buy from the firm or not. In stage 4, if the consumer buys, it observes the firm's true quality, and decides whether to buy again or not. The firm and consumer discount payoffs in this last period by $\delta$. Consumers wishing to buy in a period, buy a single unit, receiving utility $v_t$ from the good of type $t$. The unit cost of production is $c_t$. Assume (A1) $v_H > v_L > c_H > c_L$, (A2) $v_H - c_H > v_L - c_L$ and (A3) $\delta < (c_H - c_L) / (v_H - c_H)$.

Consider first the game with exogenous quality. Milgrom and Roberts (1986) use standard refinements of sequential equilibria to focus on the least-cost separating equilibrium outcome, which is also our focus here. In the resulting refined equilibrium, a high quality firm sets its price at the monopoly level $v_H$ and advertises at the minimum level (denoted $A^*$) required to ensure that the low quality firm will not want to mimic it. If the low quality firm chooses $(P, A) = (v_H, A^*)$ in stage 2, it will make a sale in stage 3 but not a repeat sale since its price exceeds its true quality $v_L$. Its profit will therefore be $v_H - c_L - A^*$. Alternatively, if the firm prices at $v_L$, consumers will be willing to buy from it in both stages 3 and 4, and so if it chooses no wasteful advertising, it can obtain a profit of $(1 + \delta)(v_L - c_L)$. Therefore, to prevent the low quality firm from wanting to mimic it, the high quality firm will choose $A^* = (v_H - c_L) - (1 + \delta)(v_L - c_L)$. (A1)-(A3) imply $A^* > 0$, so a high quality firm can indeed signal it is high quality, but this requires a positive level of dissipative advertising.

Contrast this to a setting where quality is determined endogenously at stage 1. In the reordered game in which price and advertising are chosen in stage 1 and quality in stage 2, consumers will never buy if $P > v_L$ since (A3) implies the firm always has an incentive to choose low quality. That is, if the firm sets $P > v_L$ and consumers expect the firm to produce high quality, the firm obtains $(1+\delta)(P - c_H) - A$ if it chooses high quality and $P - c_L - A$ if it chooses low quality. Under (A3), the firm obtains higher profit from producing low quality. The problem is second period profits are discounted too much. Alternatively, if $P \leq v_L$, the firm will sell in both periods regardless of its quality, and so it will again prefer to produce low quality. The firm therefore maximizes its profit by choosing $P = v_L$ in the first stage, with $A = 0$. Consumers will purchase from the firm in both stages. This is the unique $RI$-equilibrium outcome in the original game. Advertising (or more generally, burning money) is like a sunk cost in this setting and does not change a firm's incentives to supply quality. Burning money does not convince consumers the monopolist is high quality because if it did, the monopolist could do even better choosing low quality but still advertising by the same amount, thereby facing lower costs.[5]

Advertising can play a "signaling" role in the endogenous quality setting once it has a demand expanding effect. For example, advertising that increases the likelihood a buyer will consider a repeat purchase (i.e. advertising activates the recollection of past purchase experiences, as in Nelson 1974, p. 734) increases the payoff to choosing high quality since it strengthens the repeat purchase effect, potentially enabling an equilibrium with high quality to be restored in the above model. Here we provide a formal model of prices and advertising as signals which captures Nelson's particular "story" of advertising.

To do so, we adjust the model above to allow that in stage 4, the consumer only makes the decision of whether to buy again or not with probability $\phi(A)$. This probability is assumed to be strictly increasing in the firm's advertising expense, capturing the idea of Nelson that advertising activates the recollection of a past purchase decision. We modify (A3) so that $\delta\phi(0) < (c_H - c_L)/(v_H - c_H)$, which means that without any advertising, the firm would never choose high quality. In addition we assume (A4) $\phi''(A) < 0$ and (A5) $\delta\phi'(0)(v_H - c_H) < 1$. (A4)-(A5) ensure that in case the firm's quality is known to consumers, the firm would not want to advertise at all, since the marginal benefit of advertising is always less than the constant marginal cost of advertising. Thus, in this model, advertising only arises when it is needed for signaling purposes.

Consider the reordered version of this new endogenous quality game. With $P > v_L$, the firm now obtains $(1 + \delta\phi(A))(P - c_H) - A$ if it chooses high quality and $P - c_L - A$ if it chooses low quality. Define

$$A^*(P) = \phi^{-1}\left(\frac{c_H - c_L}{\delta(P - c_H)}\right).$$

For $A \geq A^*(P)$, $(1 + \delta\phi(A))(P - c_H) \geq P - c_L$, so the unique equilibrium in the subgame

[5] Likewise, in a static model in which costs are increasing in quality and no consumers are informed of quality, a high price will not convince consumers the monopolist is high quality because if it did, the monopolist could do even better with the same high price but choosing low quality, thereby enjoying lower costs.

is that the firm will indeed prefer to choose high quality and consumers will buy from it. Alternatively, if $A < A^*(P)$, the unique equilibrium in the subgame is that consumers will not buy from the firm and it will obtain no profit. In case, $P \leq v_L$, then the firm obtains $(1 + \delta\phi(A))(P - c_H) - A$ if it chooses high quality and $(1 + \delta\phi(A))(P - c_L) - A$ if it chooses low quality, regardless of the level of advertising. The unique equilibrium in the subgame is therefore one in which the firm produces low quality and consumers always buy from it.

Turning to the first stage of the reordered game in which the firm selects $(P, A)$, (A4)-(A5) imply the firm will optimally either choose $(P, A) = (v_L, 0)$ and obtain

$$(1 + \delta\phi(0))(v_L - c_L) \tag{12}$$

or set $(P, A) = (v_H, A^*(v_H))$ and obtain

$$\begin{aligned} & (1 + \delta\phi(A^*(v_H)))(v_H - c_H) - A^*(v_H) \qquad (13) \\ = \quad & v_H - c_L - A^*(v_H). \end{aligned}$$

Thus, in this setting, advertising will be used to signal high quality, whenever (13) exceeds (12), with the firm producing the high quality product and charging a price of $v_H$ in this case. Interestingly, the level of advertising is higher when firms face less discipline from repeat purchases ($\delta$ is lower). However, if $\delta$ is too low, then the firm will prefer not to advertise at all.

## 2.4 Corporate finance and costly collateral pledging

We take the benchmark model of adverse selection and signaling from Tirole (2006, p. 253) and adapt it so the "type" of the borrower is endogenously determined. In the game we consider, a borrower (entrepreneur) first chooses whether to exert effort or not which costs $K$. If the borrower exerts effort, then the borrower can invest $I$ in a project which will have a payoff of $R > 0$ with probability $0 < p < 1$ and 0 otherwise. If the borrower does not exert effort, then the same project will have a payoff of $R$ with probability $0 < q < p$, and 0 otherwise. Following its choice of effort, but before observing the outcome of the project, the borrower proposes a contract which is denoted $(R_b, C)$. The borrower chooses how much it will pay the lender (investor) in the case of success (denoted $R_b$) and also how much collateral $C \geq 0$ it will transfer to the lender in case of failure. The lender values this collateral at $\beta C$ where $0 < \beta < 1$. The lender observes the contract offer but not the borrower's effort and must decide whether to fund the project by investing $I$. The lender requires the recovery of $I$ in expectation to fund the investment.

To make the problem interesting we assume $K$ satisfies

$$\left(1 - \frac{q}{p}\right)(pR - I) < K < pR - I. \tag{14}$$

The right-hand side inequality ensures that in a full information setting, the borrower's effort and the lender's investment can deliver a positive surplus. The left-hand side inequality ensures that if the lender is willing to invest without any collateral based on the expectation that the borrower will exert effort (so $pR_b = I$), then the borrower will not actually want to exert effort (since $q(R - R_b) > p(R - R_b) - K$). We also assume $(p - q) R > K$, so exerting effort is efficient. Finally, we assume $qR < I$, so if the borrower does not exert effort, then the surplus from the project is negative.

Consider now the reordered game. In this case, the contract is chosen first, and then the borrower decides on effort and the lender decides whether to accept the contract. If the contract $(R_b, C)$ satisfies

$$p(R - R_b) - (1 - p) C - K \geq q(R - R_b) - (1 - q) C \tag{15}$$

$$pR_b + (1 - p) \beta C \geq I, \tag{16}$$

then there is a unique equilibrium in the subgame that follows in which the borrower exerts effort and the lender accepts the contract. For any $(R_b, C)$ not satisfying one or both of these constraints, one can show using the assumptions above that in the unique equilibrium in the resulting subgame, the borrower can expect to get at best zero profit (either because the borrower exerts no effort so the surplus from the project is negative, or the lender will reject the contract even if the lender expects the borrower to exert effort, or both).

The candidate for the best first-stage choice of $(R_b, C)$ is therefore the one maximizing the payoff to the borrower, $\pi = p(R - R_b) - (1 - p) C - K$, subject to the constraints (15)-(16). This implies

$$\begin{aligned} C^* &= \frac{pK - (p - q)(pR - I)}{((1 - \beta) p + \beta)(p - q)} > 0 \\ R - R_b &= \frac{\beta (1 - p) K + (pR - I)(p - q)}{((1 - \beta) p + \beta)(p - q)} > 0, \end{aligned}$$

and the constraints are binding. The resulting borrower's equilibrium payoff is

$$\pi^* = \frac{(pR - I)(p - q) - (p(1 - q) - \beta q (1 - p)) K}{((1 - \beta) p + \beta)(p - q)},$$

which is positive given

$$(pR - I)(p - q) - (p(1 - q) - \beta q(1 - p)) K > (pR - I - K)(p - q) > 0,$$

where the last inequality follows from the right-hand side of (14). Note the borrower will put up positive collateral (i.e. $C^* > 0$) and will pay the lender less than the full-information among (i.e. $R_b^* < \frac{I}{p}$). These results follow from the left-hand side of (14).

Conceptually, (15) is subtly different from the usual no-mimicking condition when the borrowers' types are determined by nature. In the exogenous signaling game, the condition was that the bad borrower did not want to try to mimic by offering the same contract that a good borrower would offer. In the separating equilibrium it should be better off offering its full-information contract and revealing it is bad. In the analysis above, (15) requires that the borrower should not be better off avoiding effort in the first place but offering the same contract (assuming the investor still thinks the borrower is "good") versus choosing effort and so actually being "good". Both are no-mimicking like constraints, except the comparison is different. In the exogenous signaling game the payoff is determined by ensuring that the bad borrower is revealed as bad; in the endogenous signaling game the payoff is determined by ensuring that the borrower chooses to be "good". This means the factors which influence the contract, or the way they do, can be different.

Turning to comparative statics, we have

$$\frac{dC^*}{dp} = -\frac{(\beta + p(1-\beta))Kq + (1-\beta)(p-q)((p-q)I + Kp) + \beta R(p-q)^2}{(\beta + p(1-\beta))^2 (p-q)^2} < 0,$$

and

$$\frac{dC^*}{dq} = \frac{pK}{((1-\beta)p + \beta)(p-q)^2} > 0,$$

which are the opposite of the result in Tirole (2006) in which the borrower's type is determined exogenously. When the prospects improve for being high quality (or deteriorate for being low quality), less collateral will be posted. This can be explained. As $p$ increases (or $q$ decreases), then the alternative of not exerting effort becomes less profitable, so the borrower does not need to distort the contract so much in order that it will want to carry out effort. Clearly, the results also imply, the more costly is effort ($K$ increases), the more collateral will be posted, which is consistent with a more severe moral hazard problem resulting in more collateral being posted.

# 3 Appendix to Section 5

This section provides the technical materials, the reordering algorithm and the guide to reordering which were referred to in Section 5 of the main paper.

## 3.1 Preliminaries

This section provides some of the notation and concepts needed for subsequent sections. We use the definition of extensive-form games presented by Hart (1992), originally due to Kuhn (1953). An extensive-form game $G$ is a tuple $(N, (X, \preceq), P, \rho, (I^i)_{i \in N}, (u_i)_{i \in N})$, where $N = \{1, 2, \ldots, n\}$ is a set of players; $(X, \preceq)$ is a rooted tree (also called the game tree), where $X$ is a finite set of nodes and $\preceq$ is the partial order on $X$, called precedence relation; $Z \subset X$ is the set of terminal nodes; $P$ is a player partition, which partitions the set of non-terminal nodes (each of the non-terminal nodes is also called a "move") $X \setminus Z$ into $n+1$ cells, $P^0$, $P^1$, $\cdots$, $P^n$; $\rho$ specifies probability distributions over states at each of nature's moves (each element of $P^0$); $I^i$ is an information partition, which partitions $P^i$ into information sets for each player $i$; and

$u_i : Z \to R$ is the payoff function for each player $i$.

For any player $i \in N$, all nodes in an information set $I^i_j$ (where $j$ is the index of a particular information set for player $i$) have the same number of outgoing branches, and every path in the tree from the root to a terminal node can cross each $I^i_j$ at most once. Each of these branches at each of the information sets is called an action, and the set of actions available at each information set $I^i_j$ is called an action set, and denoted by $A(I^i_j)$. We assume the action sets are finite, requiring that they contain at least two actions and that no action is redundant.[6]

Let $x$ be a typical element in the set of nodes $X$. When $x_1 \preceq x_2$, we say that $x_1$ weakly precedes $x_2$ or equivalently $x_2$ weakly succeeds $x_1$ ($x_1$ is a weak predecessor of $x_2$ and $x_2$ is a weak successor of $x_1$). When $x_1 \preceq x_2$ and $x_1 \neq x_2$, we write $x_1 \prec x_2$ and say that $x_1$ precedes $x_2$ or equivalently $x_2$ succeeds $x_1$ ($x_1$ is a predecessor of $x_2$ and $x_2$ is a successor of $x_1$). We say that $x_2$ is a direct successor of $x_1$ (and $x_1$ is a direct predecessor of $x_2$) if $x_1 \prec x_2$ and $x_1 \prec x_3 \preceq x_2$ implies $x_3 = x_2$, and write $x_2 \in s(x_1)$ and $x_1 = p(x_2)$. We also say that an information set $I_j$ precedes another information set $I_k$ or equivalently $I_k$ succeeds $I_j$ if for each node $x_2 \in I_k$ there exists a node $x_1 \in I_j$ such that $x_1 \prec x_2$. We refer to a player's information sets which do not have any preceding information sets among the player's information sets as the player's first information sets.

A (behavior) strategy $\pi_i$ for a player $i \in N$ is a function that associates each of the player's information sets with a probability distribution on the player's action set, that is $\pi_i(I^i_j) \in \Delta A(I^i_j)$, where $\Delta A(I^i_j)$ is the space of probability distributions on the action set $A(I^i_j)$. We call $\pi_i(I^i_j)$ player $i$'s local strategy at $I^i_j$. A strategy profile $\pi$ is a list $(\pi_i)^n_{i=1}$. We call the probability distribution on the terminal nodes that is induced by a strategy profile the outcome of the game associated with the strategy profile. We also denote the set of all strategies for player $i$ by $\Pi_i$ and the set of all strategy profiles by $\Pi$. We also say that player $i$'s strategy $\pi_i \in \Pi_i$ is a pure strategy if probability distributions at all of its information sets are degenerate, and denote the set of all pure strategies for player $i$ by $S_i$ and the set of all pure-strategy profiles by $S$.

We denote the above class of extensive-form games with perfect recall by $\Gamma$. Each extensive-form game in $\Gamma$ induces its own normal-form game $(N, (S_i)_{i \in N}, (\tilde{u}_i)_{i \in N})$, where $N$ is the set of players, $S_i$ is the set of pure strategies for each player $i$, and $\tilde{u}_i$ is a payoff function for each player $i$, induced by $u_i$. The induced payoff function $\tilde{u}_i$ (in the normal-form game) associates each pure-strategy profile $s \in S$ with payoffs while the payoff function $u_i$ (in the extensive-form game) associates each terminal history with payoffs. We say that two pure strategies for a player are equivalent if they yield the same payoffs to all players for any combination of the other players' pure strategies. By taking only one representative pure strategy from each set of equivalent pure strategies, we obtain the reduced normal-form game. For example, in the normal-form game of the extensive-form game in panel (a) of Figure 1 of the main paper, the sender has eight pure strategies, whereas in the reduced normal-form game, the sender has only four pure strategies.

[6] We say that an action at an information set is redundant if there is another action in the action set which leads to the same payoffs to all players for any combination of subsequent actions.

## 3.2 Class of endogenous signaling games

In this section, we explain further what we mean by a sender's actions or combinations of actions being partially observed by a receiver and provide a formal definition of the property in item 4 in the definition of the class of endogenous signaling games $\Gamma_S$ (in Section 5.1 of the main paper).

Suppose a sender has more than one information set in the signaling stage. Then "partial observability" means that among the combinations of actions taken at the sender's information sets, more than one combination of actions, but not all combinations of actions are taken in the paths through the receiver's particular information set. Consider the endogenous signaling game illustrated in panel (a) of Figure 1 of the main paper, in which the sender's four combinations of actions $(I, L)$, $(I, H)$, $(N, L)$, and $(N, H)$ are partially observed by the receiver in the reaction stage (*i.e.* the combination of actions $(I, L)$ and $(N, L)$ are partially observed by the receiver at one of its information sets, and the combination of actions $(I, H)$ and $(N, H)$ are partially observed by the receiver at its other information set).

Suppose a sender has only one information set in the signaling stage. Then partial observability means that among the actions taken at the sender's information set, more than one action, but not all actions are taken in the paths through the receiver's particular information set. Suppose we modify the above game such that the sender makes the two choices jointly at one move, *i.e.* choosing among $IL$, $IH$, $NL$, and $NH$. Then the sender's four actions $IL$, $IH$, $NL$, and $NH$ are partially observed by the receiver in the reaction stage (*i.e.* the actions $IL$ and $NL$ are partially observed by the receiver at one of its information sets, and the actions $IH$ and $NH$ are partially observed by the receiver at its other information set).

We require that each player's information on another player's previous actions should be independent of its information on a third player's previous actions in item 4 of the definition of the class of games $\Gamma_S$. Requiring this property is equivalent to requiring the *observable-deviators* property, which is due to Battigalli (1996, 1997).

Definition (Observable Deviators) For any information set $I^i_j$ of player $i$'s, let $S(I^i_j)$ denote the set of strategy profiles inducing a play that reaches a node in $I^i_j$. Let $S_1(I^i_j), S_2(I^i_j), \ldots, S_n(I^i_j)$ be its projections on each player's strategy sets $S_1, S_2, \ldots, S_n$. Then a game in $\Gamma$ has *observable deviators* if for all $i \in N$ and for all $I^i_j \in I^i$, $S(I^i_j) = S_1(I^i_j) \times S_2(I^i_j) \times \ldots \times S_n(I^i_j)$. ■

## 3.3 Reordering algorithm

In this section we show the algorithm of reordering the original game, which makes the relevant non-singleton information sets definite. The reordering involves a combination of the coalescing of moves and the interchange of moves (see Thompson (1952) for the definition of these two transformations), and is based on the receivers' information on the senders' actions.

The algorithm requires some additional definitions. For any game in $\Gamma_S$, we refer to each sender's action (in case it has only one information set in the signaling stage) or combination of actions (in case it has more than one information set in the signaling stage) taken in the signaling stage as its *composite action.* In the illustrative example in Section 3 of the main paper, the sender has four composite actions $IL$, $IH$, $NL$, and $NH$, and in the game in panel (a) of Figure 1, each sender has four composite actions $a_i r_i$, $a_i n_i$, $a'_i r_i$, and $a'_i n_i$.

(a) original game　　　　　　(b) reordered game

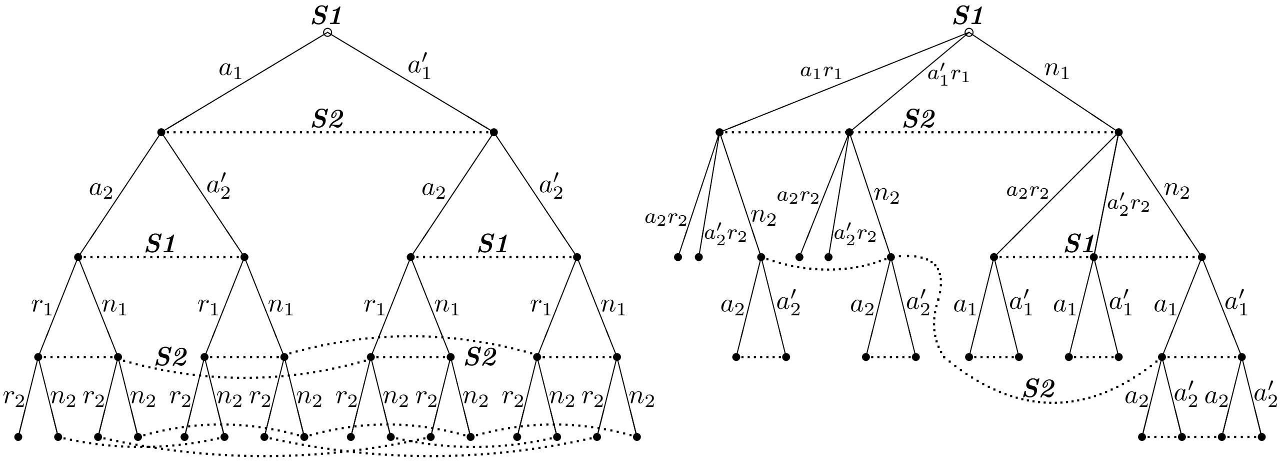


Figure 1: Endogenized observability

Consider the collection of a receiver's first information sets in the reaction stage in a game in $\Gamma_S$. Since games in $\Gamma_S$ have the observable-deviators property, each of the receiver's first information sets can be expressed as a Cartesian product of sets where each component shows the set of possible composite actions taken by each sender (or the set of possible actions taken by each player who is not a sender) to reach the information set. We collect from the receiver's first information sets, the component (of the Cartesian product) showing the set of possible composite actions taken by a particular sender[7], and call the collection the receiver's *information partition with respect to the sender* and each element of the partition an *information set with respect to the sender.*[8] As an example, consider the game in panel (a) of Figure 1, which is a game in $\Gamma_S$. In the game, all nine of the receiver's ($R$'s) information sets are its first information sets. Four of them are singleton, and the other five are non-singleton information sets. Each of the information sets can be expressed as a Cartesian product, as shown in Table 2. Collecting the components for each sender ($S_1$ and $S_2$), we obtain the receiver's information partition with respect to each sender $\{\{a_i r_i\}, \{a'_i r_i\}, \{a_i n_i, a'_i n_i\}\}$.

[7]We ignore the receiver's information partition with respect to players other than senders because they are either a singleton partition (in case all the player's actions taken in the signaling stage are observed by the receiver) or a trivial partition (in case all the player's actions taken in the signaling stage are unobserved by the receiver), and so the reordering of the information sets belonging to these players is not necessary.

[8]For the sake of expositional simplicity, we proceed assuming every terminal history of the game includes at least one action by each receiver. Otherwise, the affected receiver's information partition with respect to a sender may not cover all of the sender's composite actions. In this case, we just need to complete the partition simply by adding a dummy information set containing all of the sender's

| Receiver's first information sets | expressed as Cartesian products |
|---|---|
| $\{(a_1r_1, a_2r_2)\}$ | $\{a_1r_1\} \times \{a_2r_2\}$ |
| $\{(a_1r_1, a'_2r_2)\}$ | $\{a_1r_1\} \times \{a'_2r_2\}$ |
| $\{(a'_1r_1, a_2r_2)\}$ | $\{a'_1r_1\} \times \{a_2r_2\}$ |
| $\{(a'_1r_1, a'_2r_2)\}$ | $\{a'_1r_1\} \times \{a'_2r_2\}$ |
| $\{(a_1r_1, a_2n_2), (a_1r_1, a'_2n_2)\}$ | $\{a_1r_1\} \times \{a_2n_2, a'_2n_2\}$ |
| $\{(a'_1r_1, a_2n_2), (a'_1r_1, a'_2n_2)\}$ | $\{a'_1r_1\} \times \{a_2n_2, a'_2n_2\}$ |
| $\{(a_1n_1, a_2r_2), (a'_1n_1, a_2r_2)\}$ | $\{a_1n_1, a'_1n_1\} \times \{a_2r_2\}$ |
| $\{(a_1n_1, a'_2r_2), (a'_1n_1, a'_2r_2)\}$ | $\{a_1n_1, a'_1n_1\} \times \{a'_2r_2\}$ |
| $\{(a_1n_1, a_2n_2), (a_1n_1, a'_2n_2), (a'_1n_1, a_2n_2), (a'_1n_1, a'_2n_2)\}$ | $\{a_1n_1, a'_1n_1\} \times \{a_2n_2, a'_2n_2\}$ |
| Information partition with respect to | |
| sender 1 | $\{\{a_1r_1\}, \{a'_1r_1\}, \{a_1n_1, a'_1n_1\}\}$ |
| sender 2 | $\{\{a_2r_2\}, \{a'_2r_2\}, \{a_2n_2, a'_2n_2\}\}$ |

Table 2: Receiver's first information sets

These two concepts are similar to a player's "information partition" and "information sets" in their standard usage, except that they are projections of their standard counterparts to the space of a particular sender's composite actions. As usual, a singleton information set implies that the receiver knows perfectly which composite action the sender has taken whereas a non-singleton information set implies that it is confused about which composite action the sender has taken. We call the Cartesian product of the receiver's information partitions, each of which is with respect to each sender, the receiver's *information partition with respect to all senders.* This partition summarizes the receiver's information on all the senders' actions and has an important role in the reordering algorithm. In the above example, the receiver's information partition with respect to all senders is $\{\{a_1r_1\}, \{a'_1r_1\}, \{a_1n_1, a'_1n_1\}\} \times \{\{a_2r_2\}, \{a'_2r_2\}, \{a_2n_2, a'_2n_2\}\}$.

Suppose there are $m$ senders. We fix an order of the $m$ senders so that the senders are labeled $1, 2, \ldots, m$. We categorize receivers based on their information partitions with respect to all senders (hereafter we simply write " information partition" omitting "with respect to all senders"). For each of the different information partitions, we construct an extensive-form game, where the information sets in the signaling stage are rearranged according to the following algorithm and the payoff functions are inherited from the original game up to the relabeling of actions. We call the extensive-form game constructed this way the *reordered game* with respect to the associated information partition, and each of the receivers whose information partition is used to construct the reordered game an *associated receiver.*

1. Sort information sets in the signaling stage as follows: information sets of the players all of whose actions are observed by the associated receivers (if any), sender 1's information sets, sender 2's information sets, …, sender $m$'s information sets, information sets of the players all of whose actions are unobserved by the associated receivers (if any).

2. For each sender who has more than one information set, coalesce them into one informa-

uncovered composite actions.

tion set, which we call its *composite information set*, such that the actions taken there are its composite actions. For each sender having only one information set, consider the information set as its composite information set and each of its actions as a composite action for the next step.

3. For each sender, decompose the composite information set from step 2 into a first information set and succeeding information set(s) such that (i) each available action at the sender's first information set leads to each distinct cell (*i.e.* information set) of the associated receivers' information partition with respect to the sender, and (ii) each available action at the sender's succeeding information sets[9] leads to each distinct element of the cell.[10]

4. Reorder the senders' first and succeeding information sets according to the order of senders as follows: sender 1's first information set, sender 2's first information set, ..., sender $m$'s first information set, sender 1's succeeding information sets (if any), sender 2's succeeding information sets (if any), ... , sender $m$'s succeeding information sets (if any).

We use the "interchange of moves" iteratively in steps 1 and 4 and the "coalescing of moves" for each sender in each of steps 2 and 3. As a result, each of the reordered games shares the same reduced normal form as the original game. Note that for endogenous signaling games with only one sender, steps 1 and 4 are not necessary. Even with multiple senders, step 4 is not essential, but it ensures the reordered game is played in a natural way such that senders choose their observed actions first simultaneously and then their unobserved actions simultaneously. Step 3 makes sure that all actions available at each sender's first information set are observed and all actions available at each sender's succeeding information sets are unobserved by the associated receivers. For this reason, all associated receivers' non-singleton information sets become definite in the reordered game, which is at the heart of the proof of Proposition 4.

Suppose in the illustrative example in Section 3 of the main paper the sender makes the two choices jointly, *i.e.* chooses among $IL$, $IH$, $NL$, and $NH$ at one move. Then each of the four actions are partially observed by the receiver. The receiver's information partition (with respect to the sender) is $\{\{IL, NL\}, \{IH, NH\}\}$. Since there is only one sender, steps 1 and 4 are not necessary. We consider the sender's information set as its composite information set (step 2). Step 3 means that we decompose the sender's composite information set to its first information set where it chooses between $\{IL, NL\}$ and $\{IH, NH\}$ (*i.e.* between $L$ and $H$), which are observed by the receiver, and its succeeding information sets where it chooses between $I$ and $N$, which are unobserved by the receiver. Applying the algorithm, we obtain the same reordered game as in panel (b) of Figure 1 of the main paper.

[9]Note that among the same sender's succeeding information sets obtained in step 3, no information set precedes another information set.

[10]In case an action available at the sender's first information set leads to the associated receivers' singleton information set in their information partition with respect to the sender, there is no succeeding information set following the action.

Consider the game in panel (a) of Figure 1. Recall that the receiver's information partition with respect to each sender is $\{\{a_i r_i\}, \{a'_i r_i\}, \{a_i n_i, a'_i n_i\}\}$. Step 1 of the algorithm means that we interchange sender 2's first information set and sender 1's succeeding information sets. Step 2 means that we construct the composite information set for each sender, where it has four composite actions available: $a_i r_i$, $a'_i r_i$, $a_i n_i$, and $a'_i n_i$. Step 3 means that we decompose each sender's composite information set to its first information set and its succeeding information set. At the first information set, it chooses an action leading to one of the three cells of $\{\{a_i r_i\}, \{a'_i r_i\}, \{a_i n_i, a'_i n_i\}\}$. In case it has chosen $\{a_i n_i, a'_i n_i\}$ ("not reveal") at the first information set, the succeeding information set follows, where it chooses an action leading to one of the elements of $\{a_i n_i, a'_i n_i\}$, *i.e.* between the unrevealed actions $a_i$ and $a'_i$. Note that the actions at the senders' first information sets are observed and the actions at their succeeding information sets are unobserved by the receiver in the reordered game. Step 4 means that we reorder the senders' information sets such that senders choose their observed actions simultaneously, then unobserved actions simultaneously.

In case the receivers have different information regarding the sender's actions, in general we need to construct a reordered game for each receiver, following the reordering above. This case is illustrated with the example of private contracting in Section 5.4 in the main paper. However, one reordered game may be sufficient if the different information partitions can be combined to form a linearly-ordered set under the usual partial order. For example, consider a game with one sender and three receivers. The sender chooses one out of eight actions $a, b, \ldots, h$ at its single move. Then three receivers move simultaneously, having observed the sender's action partially, according to the following information partitions:

$$\begin{aligned}
\text{Receiver 1:}\quad I^1 &= \{\{a,b,c,d\},\{e,f\},\{g,h\}\},\\
\text{Receiver 2:}\quad I^2 &= \{\{a,b\},\{c,d\},\{e,f,g,h\}\},\\
\text{Receiver 3:}\quad I^3 &= \{\{a\},\{b\},\{c,d\},\{e,f\},\{g,h\}\}.
\end{aligned}$$

The three information partitions can be combined to form a linearly-ordered set

$$\{\{\{a,b,c,d\},\{e,f,g,h\}\},\{\{a,b\},\{c,d\},\{e,f\},\{g,h\}\},\{\{a\},\{b\},\{c,d\},\{e,f\},\{g,h\}\}\}.$$

Then it is sufficient to construct just one reordered game such that the sender chooses between $\{a,b,c,d\}$ and $\{e,f,g,h\}$, followed by another choice between $\{a,b\}$ and $\{c,d\}$ (if the previous choice was $\{a,b,c,d\}$) or between $\{e,f\}$ and $\{g,h\}$ (if the previous choice was $\{e,f,g,h\}$), followed by the last choice between the two elements in each of the two-element sets.

In the described algorithm above, we implicitly assumed that the associated receivers' information on the senders' actions does not change (*i.e.* the information partition does not get refined in the reaction stage) for the simplicity of the exposition. If the associated receivers' information changes, but independently of their choices, constructing a single reordered game would be sufficient. In this case, steps 1 and 2 above still apply, but step 3 should be more involved such that the reordering may reflect the sequence of information refinement. For example, suppose the original game is such that there is a single sender with five available actions at its single move, $\{a, b, c, d, e\}$, and the associated receivers' information partition on the sender's actions gets refined as it moves sequentially in the reaction stage, $\{\{a, b, c\}, \{d, e\}\}$ $\to \{\{a, b\}, \{c\}, \{d, e\}\} \to \{\{a\}, \{b\}, \{c\}, \{d, e\}\}$. Then the reordered game should be such that the sender chooses between $\{a, b, c\}$ and $\{d, e\}$, followed by another choice between $\{a, b\}$ and $c$ (if the previous choice was $\{a, b, c\}$) or between $d$ and $e$ (if the previous choice was $\{d, e\}$), followed by the last choice between $a$ and $b$ (if the previous choice was $\{a, b\}$). Suppose the change in the associated receivers' information depends on their choices. In case the different sequences of information refinement are compatible, one reordered game can still cover them all. In case the different sequences of information refinement are not compatible, we need to construct more than one reordered game, one for each incompatible sequence of information refinement. The different sequences of information refinement are (in)compatible if the information partitions in different sequences can(not) be combined to form a linearly-ordered set under the usual partial order. For example, the different sequences of information refinement are compatible if the receivers can discover different senders' unobserved actions by choosing different respective actions; and the different sequences of information refinement are incompatible if one sequence is $\{\{a, b, c\}, \{d, e\}\} \to \{\{a, b\}, \{c\}, \{d, e\}\}$ while another sequence is $\{\{a, b, c\}, \{d, e\}\}$ $\to \{\{a\}, \{b, c\}, \{d, e\}\}$.

## 3.4 A guide to reordering

The previous algorithm explains how to reorder endogenous signaling games in general. We have explained how to reorder some specific endogenous signaling games at various points in the main paper and in this online appendix. In this section we summarize these specific examples, provide some additional guidance on reordering a game with competing firms, and explain how to reorder games in which receivers move in the signaling stage (which are outside the class of endogenous signaling games defined in Section 5.1 of the main paper).

**Summary of existing cases:** For games in which a single sender moves in stage 1, and a single receiver (or multiple receivers who all share the same information regarding the sender's actions) move(s) in stage 2, we described the appropriate reordering at the start of Section 4 in the main paper. Figure 1 in the main paper illustrates with a particular example. We also discussed a number of applications of such games in Section 4.2 and provided the analysis for these applications in Section 2 of this appendix. The limit-pricing application illustrates the point that we can trivially allow for random choices by nature provided they do not arise between the sender's moves that need to be reordered.

This same reordering applies if some players observe all of the sender's actions or never observe the sender's actions, or if there are some other players (they could be competing with the single sender) whose actions are fully observed or never observed. We can also add extra stages after the reaction stage where all the previous moves become common knowledge. In this case, a practical approach is to identify the equilibrium of the continuation games and embed the equilibrium payoffs in the original payoff functions.

This same reordering also works if in the original game the sender chooses multiple actions at one move, as was explained in Section 5.2.2 of the main paper.

So far, in this section, we have assumed that if there are multiple receivers, the receivers all share the same information regarding the sender's actions. The case where receivers have different information regarding the sender's actions and yet the reordering only requires a single reordered game is discussed in the second-to-last paragraph of the previous section, where we explain how to do the reordering for one such game. An example of how to do the reordering in case multiple reordered games are required is given in Section 5.4 in the main paper.

For games in which the observability of a sender's or senders' first action depends on its choice of the subsequent actions, we explained how to do the reordering in Section 5.2.2 of the main paper, with the reordering illustrated in Figure 1 in this appendix.

For games in which there are multiple senders each of whom just chooses some unobserved and some observed actions in the signaling stage but there is a single receiver (or multiple receivers who all share the same information regarding the senders' actions) that move(s) in the reaction stage, we explained how to do the reordering in Section 5.2.1 of the main paper, and illustrated this in Figure 2 in the main paper. Next, we provide a further example of multiple senders that covers the case each sender chooses its unobserved and observed actions at a single move, and not all players have the same information on the senders' actions.

**Further example of multiple senders:** Consider the case of multiple firms that compete in price and quality. Suppose each firm chooses its price and quality at the same time. Suppose all consumers observe prices but only a fraction $\lambda_i$ of consumers observe quality from firm $i$, where $0 < \lambda_i < 1$. Alternatively, each consumer may observe the quality of firm $i$ with probability $\lambda_i$.

Suppose firm $i$'s profit can be represented as

$$\pi_i \left(p_i, p_{-i}; q_i, q_{-i}; D_i\right),$$

where $p_i$, $q_i$, and $D_i$ represent firm $i$'s price, quality and demand by consumers, and $p_{-i}$ and $q_{-i}$ represent vectors of prices and qualities for all other firms. Let $p = (p_i, p_{-i})$ and $q = (q_i, q_{-i})$ be the vectors of all prices and quantities, respectively. From firm $i$'s perspective, consumer demand for its product is determined by a consumer optimization problem, which depends on, in general, firm $i$'s price and quality (either actual or expected quality, depending on whether consumers observe the firm's quality or not) and equilibrium prices and qualities of all other firms. As a result, the pure-strategy *RI-equilibrium* is defined by finding the equilibrium of the reordered game, which requires for all $i$

$$\begin{aligned}
\tilde{p}_i &\in \arg\max_{p_i} \left\{ \begin{array}{c} \lambda_i \pi_i \left( p_i, \tilde{p}_{-i}; \tilde{q}_i(p_i), \tilde{q}_{-i}; \tilde{D}_i \left( p_i, \tilde{p}_{-i}, \tilde{q}_i(p_i), \tilde{q}_{-i} \right) \right) \\ + (1-\lambda_i)\, \pi_i \left( p_i, \tilde{p}_{-i}; \tilde{q}_i\left(p_i\right), \tilde{q}_{-i}; \tilde{D}_i \left( p_i, \tilde{p}_{-i}, q_i^e\left(p_i\right), \tilde{q}_{-i} \right) \right) \end{array} \right\}, \\
\tilde{q}_i(p_i) &\in \arg\max_{q_i} \left\{ \begin{array}{c} \lambda_i \pi_i ( p_i, \tilde{p}_{-i}; q_i, \tilde{q}_{-i}; \tilde{D}_i \left( p_i, \tilde{p}_{-i}, q_i, \tilde{q}_{-i} \right)) \\ + (1-\lambda_i)\, \pi_i \left( p_i, \tilde{p}_{-i}; q_i, \tilde{q}_{-i}; \tilde{D}_i \left( p_i, \tilde{p}_{-i}, q_i^e\left(p_i\right), \tilde{q}_{-i} \right) \right) \end{array} \right\} \quad \forall p_i, \\
q_i^e(p_i) &= \tilde{q}_i(p_i) \quad \forall p_i. \\
\tilde{q}_i &= \tilde{q}_i(\tilde{p}_i),
\end{aligned}$$

where $q_i^e(p_i)$ denotes consumers' belief about firm $i$'s quality $q_i$. It is useful to note that the Envelope theorem can be used to simplify the first-order conditions for this reordered game since when differentiating profit with respect to price, we can set the impact through $\tilde{q}_i(p_i)$ to zero. However, there remains the direct effect of changing price, and the signaling-commitment effect through $q_i^e(p_i)$. Having found the equilibrium of the reordered game based on the conditions above, we can take this equilibrium outcome as the equilibrium outcome of the original game. This is the *RI-equilibrium* outcome that is defined in Section 5.2.3 of the main paper.

**Receivers move in the signaling stage:** In the class of games defined in Section 5.1 of the main paper, we assumed receivers move only in the reaction stage. This made it easier to express some of our definitions and formal results. In practice, we can still make use of *Reordering Invariance* for these games. In the remainder of this section, we show how to reorder two particular examples of interest.

Suppose the sender and receiver move simultaneously in the signaling stage and/or the reaction stage. An example would be a predator's choice of price to signal its choice of cost-reducing investment in a predation game, where the rival also sets its price at the same time as the predator prior to deciding whether to exit or not. Specifically, consider the following timing of moves in the original game.

1. In stage 1a, the sender chooses $t$.

2. In stage 1b, the sender and the receiver choose $a_S$ and $a_R$ respectively and simultaneously.

3. In stage 2, the sender (having observed the receiver's choice of $a_R$) and the receiver (having observed the sender's choice of $a_S$, but not $t$) choose $b_S$ and $b_R$ respectively and simultaneously.

In stage 1b, only the sender knows its own choice of $t$. Then the reordered game has the following timing of moves:

1. In stage 1a, the receiver and the sender choose $a_R$ and $a_S$ respectively and simultaneously.

2. In stage 1b, the sender choose $t$.

3. In stage 1c, the sender (having observed the receiver's choice of $a_R$) and the receiver (having observed the sender's choice of $a_S$, but not $t$) choose $b_S$ and $b_R$ respectively and simultaneously.

Another case of possible interest is when there are multiple senders who are also receivers. An example would be competitors' choices of price or quantity to signal their choice of cost-reducing investment (unobservable by the rivals) in a competition game with two rounds of competition. This parallels the game analyzed by Mailath (1989) in which nature determined each competitor's cost, as noted in Table 1. Specifically, suppose there are two senders who are also receivers (we will call them players). Consider the following timing of moves in the original game.

1. In stage 1a, the two players choose $t_1$ and $t_2$ respectively and simultaneously.

2. In stage 1b, the two players choose $a_1$ and $a_2$ respectively and simultaneously.

3. In stage 2, having observed the other player's choice of $a_j$ but not $t_j$, the two players choose $b_i$ $(i, j \in \{1, 2\}, i \neq j)$ simultaneously.

In stage 1b, each player $i$ knows its own choice of $t_i$, but not $t_j, j \neq i$. Then the reordered game has the following timing of moves:

1. In stage 1a, the two players choose $a_1$ and $a_2$ respectively and simultaneously.

2. In stage 1b, the two players choose $t_1$ and $t_2$ respectively and simultaneously.

3. In stage 2, having observed the other player's choice of $a_j$ but not $t_j$, the two players choose $b_i$ $(i, j \in \{1, 2\}, i \neq j)$ simultaneously.

In stage 1b, each sender $i$ knows its own choice of $a_i$, but not $a_j, j \neq i$.

## 3.5 Definiteness

In this section, we provide a formal definition of definiteness (introduced in Section 5.2.3 of the main paper) using the notation in Section 3.1.

We use Kreps and Wilson's (1982) sequential equilibrium as our basic equilibrium concept for the broader class of endogenous signaling games $\Gamma_S$. A *system of beliefs* is a function $\mu : X \to [0, 1]$ such that $\sum_{x \in I^i_j} \mu(x) = 1$ for each $I^i_j \in I^i$ for all $i \in N$. An assessment is a pair $(\mu, \pi)$ consisting of a system of beliefs $\mu$ and a strategy profile $\pi$.

DEFINITION (SEQUENTIAL EQUILIBRIUM) A *sequential equilibrium* is an assessment $(\mu, \pi)$ that is both sequentially rational and consistent. ■

Following convention, we often refer to only the strategy profile, omitting the system of beliefs, as a sequential equilibrium. When there is no risk of confusion, we also use the term "equilibrium" to mean "sequential equilibrium".

Games of imperfect information in $\Gamma$ contain at least one non-singleton information set. An equilibrium requires players' beliefs to be specified at every such information set. As was noted for the illustrative example in Section 3 of the main paper, typically there are a multitude of equilibria in games in $\Gamma$ if players' beliefs are not pinned down at such information sets.

Definition (Definiteness) We call an extensive-form game without particular payoff functions a "game structure". Consider an extensive-form game in $\Gamma$ and its game structure $(N, (X, \preceq), P, \rho, (I^i)_{i \in N})$. We say that a non-singleton information set $I^i_j$ is *definite*[11] in the game structure if for any $\pi \in \Pi$, there is a unique $\mu(x)$ for all $x \in I^i_j$ such that the assessment $(\mu, \pi)$ is consistent in any game with the game structure, and *indefinite* otherwise. ■

If a non-singleton information set is definite, the consistency requirement of sequential equilibrium pins down the belief at the information set, given any strategy profile. In Lemma 2 below, we provide a necessary and sufficient condition on the game structure for a non-singleton information set to be definite.[12]

## 3.6 Proof of Proposition 4

Consider games in $\Gamma$ where there is no nature's move. For any non-singleton information set $I_j$, let $\bigwedge(I_j)$ denote the node $x \in X$ such that $[x_k \prec x_j, \ \forall x_j \in I_j]$ implies $x_k \preceq x$. We call it $I_j$'s *nearest common predecessor*. Note that the history of the game up to $\bigwedge(I_j)$ is perfectly known by the player at $I_j$. If the player is confused about the history of the game at $I_j$, the confusion is only with respect to actions taken at the information sets containing $\bigwedge(I_j)$ and succeeding nodes. We say that $I_j$ is *complete* in the game structure (see panel (a) of Figure 2) if every terminal history of the game passing through $\bigwedge(I_j)$ also passes through a node in $I_j$ (*i.e.* $\bigwedge(I_j) \prec x$ implies $\exists x_j \in I_j$ s.t. $x \preceq x_j$ or $x_j \prec x$).

We prove in Lemma 2 that if a non-singleton information set (say, $I_j$) is complete it must be definite. If $I_j$ is not complete, we say that there is *leakage* with respect to $I_j$. In general, such leakage can make $I_j$ indefinite because the strategies may assign (locally) all probability mass to actions not leading to any node in $I_j$, but there are cases in which such leakage does not lead to indefiniteness.

Suppose there is leakage with respect to $I_j$. Then there exist nodes which are successors of $\bigwedge(I_j)$ but neither weak predecessors nor successors of any node in $I_j$. We choose the smallest (with respect to the precedence relation $\preceq$) among them and call each of them a *leaked node*. We denote the set of leaked nodes with respect to $I_j$ by $l(I_j)$. We say that leakage happens at the direct predecessors of the leaked nodes.

Let $I_l$ be an information set to which the direct predecessor of a leaked node $x_l \in l(I_j)$ belongs ($p(x_l) \in I_l$). We say that a leaked node $x_l$ is an *allowed leaked node* if the following two

[11] What is definite is not the information set itself but rather the beliefs conditional on the information set for any given behavior strategy profile (provided the consistency requirement is satisfied), but we use this terminology for the sake of brevity.

[12] A singleton information set is trivially definite.

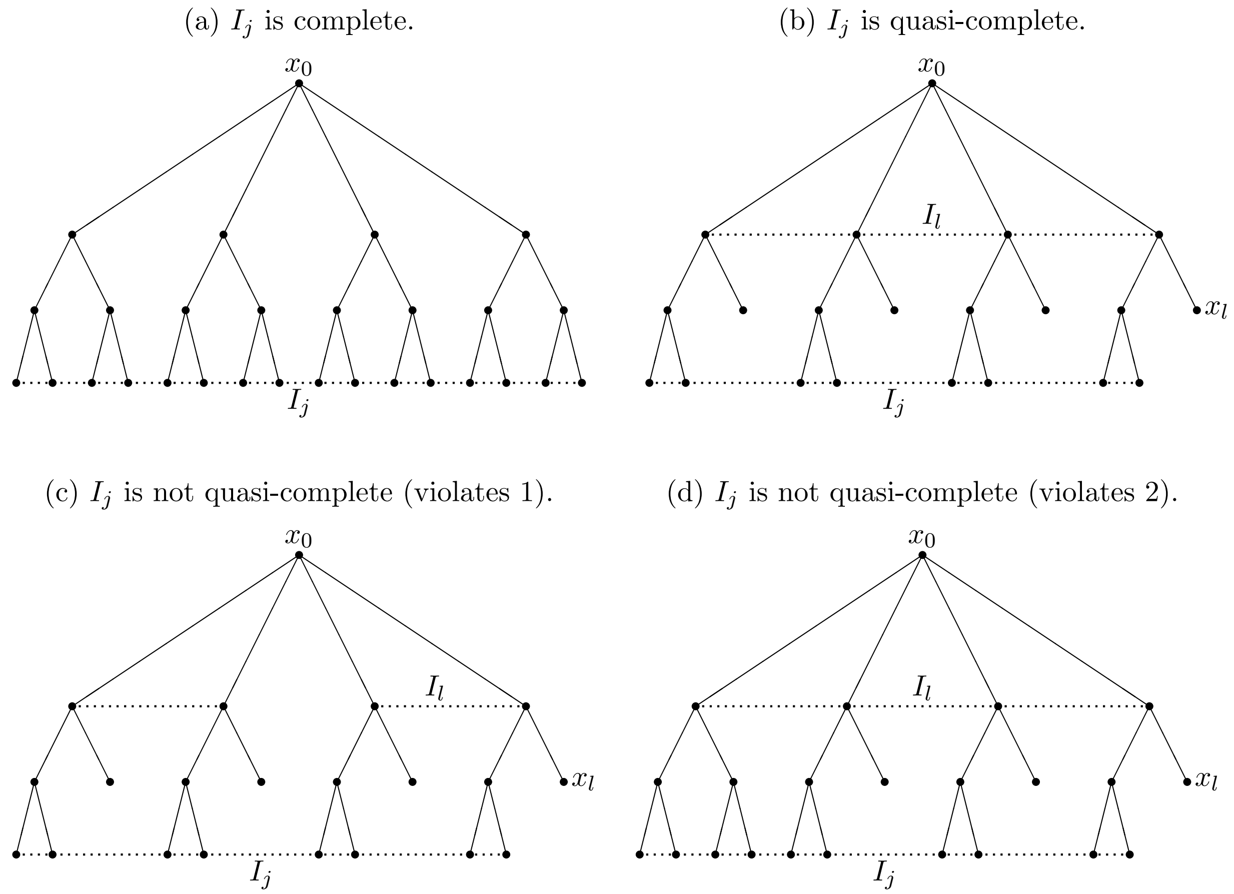


Figure 2: Completeness and quasi-completeness

conditions are satisfied for the information set $I_l$, and a *non-allowed leaked node* otherwise.[13]

1. Every path from $I_j$'s nearest common predecessor $\dot{\bigwedge}(I_j)$ to a node in the information set $I_j$ passes through the information set $I_l$.

2. Every path from $I_j$'s nearest common predecessor $\dot{\bigwedge}(I_j)$ to a node in the information set $I_j$ contains the same one action among the actions in $A(I_l)$.

We say that $I_j$ is *quasi-complete* in the game structure if it is not complete but all leaked nodes $x_l \in l(I_j)$ are allowed leaked nodes. In Figure 2, where $x_0 \equiv \dot{\bigwedge}(I_j)$, the information set $I_j$ is not quasi-complete in panel (c) because not every such path passes through the information set $I_l$ and in panel (d) because there are some such paths containing a different action in $A(I_l)$. The information set $I_j$ is quasi-complete and the node $x_l$ is an allowed leaked node in panel (b). Note that in general there can be more than one information set to which allowed leaked nodes belong although in the example illustrated in panel (b) there is only one.

Now we are ready to state our lemma.

[13]In games in $\Gamma$ where there is nature's move, if the direct predecessor of a leaked node is nature's move, then the leaked node can be considered as an allowed leaked node.

**Lemma 2** ***(Definiteness)*** *For the game structure of any game $G \in \Gamma$ where there is no nature's move, a non-singleton information set is definite if and only if it is either complete or quasi-complete.*

**Proof of Lemma 2.** (1) The proof of sufficiency is almost immediate. Suppose a non-singleton information set $I_j$ is complete or quasi-complete. Let $x_0$ be $\bigwedge(I_j)$. Then applying the local strategies at the information sets containing $x_0$ and the succeeding nodes until we reach $I_j$ (skipping the information sets that the direct predecessors of the allowed leaked nodes belong to in the quasi-complete case), we can pin down the belief at $I_j$ by the consistency requirement of sequential equilibrium.

(2) Now we prove the necessity of the condition. Suppose a non-singleton information set $I_j$ is neither complete nor quasi-complete. Then there is leakage with respect to $I_j$, and at least one leaked node is a non-allowed leaked node. Choose $x_l \in l(I_j)$ among the non-allowed leaked nodes (see Figure 3). Let $x_k = p(x_l)$. Let $x_2$ be such that $x_2 \in s(x_k)$ and $x_2 \notin l(I_j)$. Such

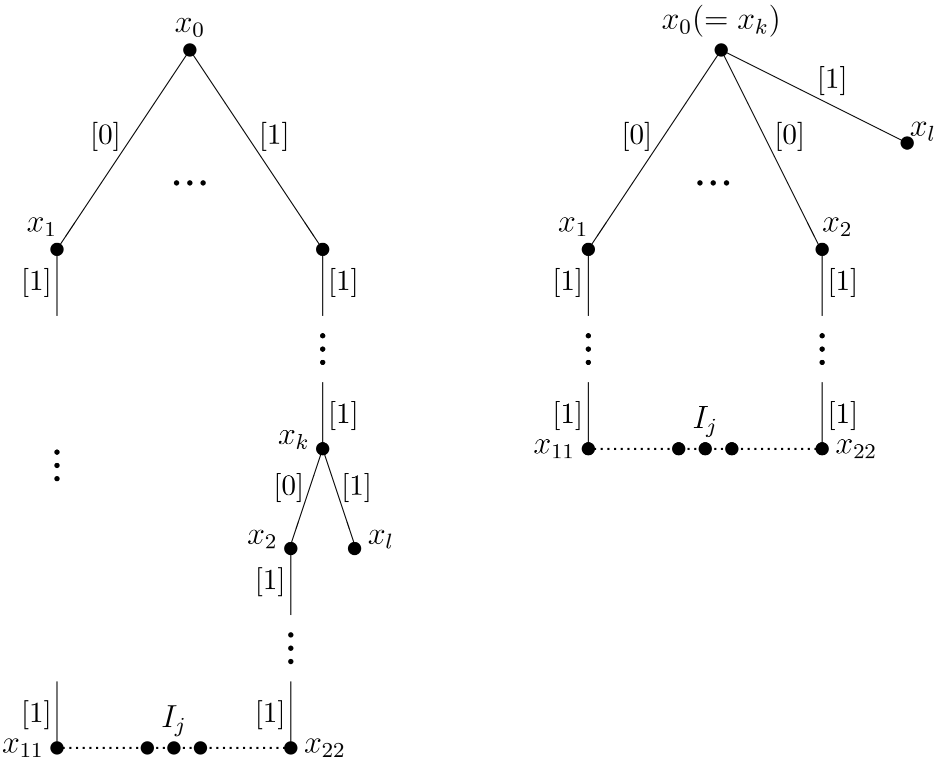


Figure 3: Reference for the proof of Lemma 2

$x_2$ exists because there are at least two nodes in $s(x_k)$ and $s(x_k) \not\subseteq l(I_j)$ by the definition of a leaked node. Let $x_1$ be such that $x_1 \not\preceq x_2$, $x_1 \in s(x_0)$, and $x_1 \notin l(I_j)$. Such $x_1$ exists for the following reason: There are at least two nodes in $s(x_0)$; Among the nodes in $s(x_0)$, exactly one node weakly precedes $x_2$; If all other nodes in $s(x_0)$ belong to $l(I_j)$ then $x_0$ cannot be $\bigwedge(I_j)$. Consider a strategy profile that assigns probability 1 to all actions chosen in the paths from $x_0$ to $x_l$, from $x_1$ to a node in $I_j$ (denoted by $x_{11}$), and from $x_2$ to a node in $I_j$ (denoted by $x_{22}$).

In this case, any belief that assigns probability $p$ to the node $x_{11}$ and $(1-p)$ to the node $x_{22}$ where $p \in [0,1]$, is compatible with the consistency requirement of sequential equilibrium. ■

Given Lemma 2 we only need to prove the non-singleton information sets of the associated receivers are either complete or quasi-complete in any reordered game. Consider any non-singleton information set (say, $I$) of any associated receiver in a reordered game. Note that all actions available at the information sets containing $\bigwedge(I)$ and its successors preceding a node in $I$ are either observed or unobserved at the non-singleton information set; that is, no action is partially observed. Given the observable-deviators property, this implies that the non-singleton information set is either complete (if all such actions are unobserved) or quasi-complete (if some such actions are observed). ■

## 3.7 Relationship with proper equilibrium

We stated in Section 5.2.4 of the main paper that for any game in $\Gamma_S$, the set of proper equilibrium outcomes is a subset of the set of *RI-equilibrium* outcomes. We show by an example that the inclusion is strict for some games in $\Gamma_S$. Consider the following canonical endogenous signaling game, which is also in the broader class $\Gamma_S$.

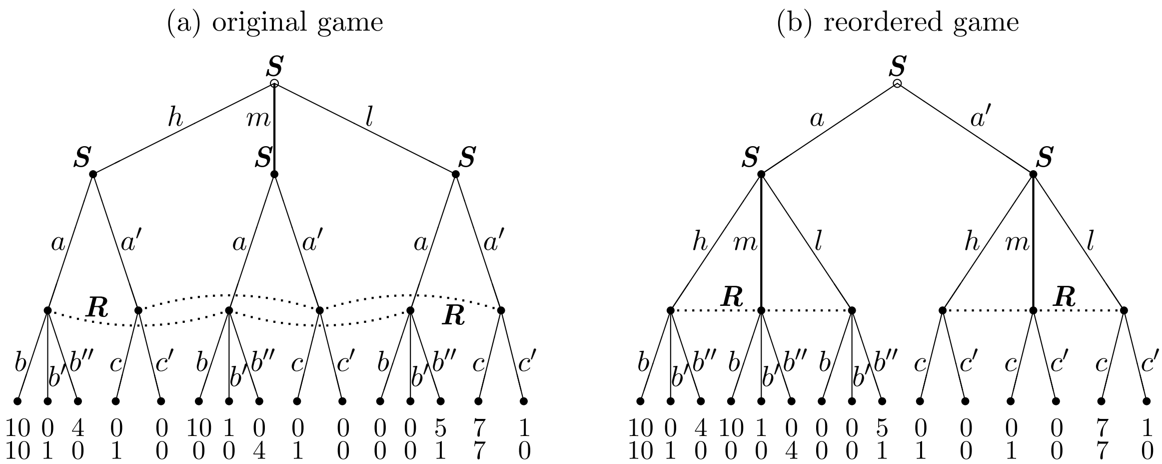


Figure 4: Original and reordered games

The reduced-normal-form game is as follows:

The following are *RI-equilibria*:

$$\begin{array}{lll} (ph+(1-p)m, a, a, a'; b, c)_{p\in[\frac{2}{7},1]} & \text{with beliefs} & \mu(h|a)=p,\ \mu(m|a)=1-p,\ \mu(l|a')=1, \\ & \text{and payoffs} & (10, 10p), \\ (l, a, a, a'; \frac{1}{11}b+\frac{10}{11}b'', c) & \text{with beliefs} & \mu(h|a)=\frac{1}{11},\ \mu(l|a)=\frac{10}{11},\ \mu(l|a')=1, \\ & \text{and payoffs} & (7,7), \\ (l, a, pa+(1-p)a', a'; b'', c)_{p\in[0,1]} & \text{with beliefs} & \mu(l|a)=1,\ \mu(l|a')=1, \\ & \text{and payoffs} & (7,7). \end{array}$$

They correspond to the following strategy profiles in the reduced-normal-form game:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | ***R*** | | | |
| | | $bc$ | $bc'$ | $b'c$ | $b'c'$ | $b''c$ | $b''c'$ |
| | $ha$ | 10, 10 | 10, 10 | 0, 1 | 0, 1 | 4, 0 | 4, 0 |
| | $ha'$ | 0, 1 | 0, 0 | 0, 1 | 0, 0 | 0, 1 | 0, 0 |
| ***S*** | $ma$ | 10, 0 | 10, 0 | 1, 0 | 1, 0 | 0, 4 | 0, 4 |
| | $ma'$ | 0, 1 | 0, 0 | 0, 1 | 0, 0 | 0, 1 | 0, 0 |
| | $la$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 5, 1 | 5, 1 |
| | $la'$ | 7, 7 | 1, 0 | 7, 7 | 1, 0 | 7, 7 | 1, 0 |

Figure 5: Reduced-normal-form game

$(pha + (1-p)ma; bc)_{p\in[\frac{2}{7},1]}$ with payoffs $(10, 10p)$,
$(la'; \frac{1}{11}bc + \frac{10}{11}b''c)$ with payoffs $(7, 7)$,
$(la'; b''c)$ with payoffs $(7, 7)$.

Among these, only $(\frac{4}{5}ha + \frac{1}{5}ma; bc)$, $(la'; \frac{1}{11}bc + \frac{10}{11}b''c)$, and $(la'; b''c)$ are proper. The set of proper equilibrium outcomes $\{\frac{4}{5}(h,a,b) + \frac{1}{5}(m,a,b), (l,a',c)\}$ is a proper subset of the set of *RI-equilibrium* outcomes $\{p(h,a,b) + (1-p)(m,a,b), (l,a',c)\}_{p\in[\frac{2}{7},1]}$.

The *RI-equilibrium* $(h,a,a,a'; b,c)$ with payoffs $(10,10)$, which is not proper in the reduced-normal-form game, is arguably the most natural one to select in the game. This equilibrium can be justified based on a forward-induction argument as follows. When the receiver observes $a$, she knows there are many equilibria that this can correspond to: some equilibria yield a payoff of 10 to the sender while the other equilibria yield a payoff less than 7 to the sender. The receiver also knows the sender could have chosen $a'$ and obtained a payoff of 7 (note that $(l,c)$ is the strictly-dominant-strategy equilibrium in the subgame following $a'$ in the reordered game). Thus, once the receiver observes that the sender has chosen $a$, she can reason that the sender must believe that they are playing one of the equilibria which yield payoff 10 to the sender since otherwise he would not have chosen $a$. In short, choosing $a$ signals to the receiver that the sender has chosen $h$ or randomized between $h$ and $m$ expecting the better payoff 10. Also note that the *RI-equilibrium* strategy profile $(ha; bc)$, which is not proper, survives the iterative elimination of weakly-dominated strategies (IEWDS) for any order of elimination, whereas all the proper equilibria do not in the reduced-normal-form game. This shows that sometimes, considering only *RI-equilibria* that are proper may be too restrictive.

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