The Optimal Public Debt Portfolios for Nine OECD Countries: A Tax Smoothing Approach

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Abstract

Within the context of currency and maturity choice, and indexation, this paper seeks to find the policy implications of a tax smoothing approach to public debt management. This objective is achieved in two ways. Firstly, the tax smoothing approach of Bohn (1990) is adapted to incorporate realistic constraints on public debt management. Secondly, this adapted methodology is applied to data on nine OECD countries. Results suggest that, for eight out of the nine countries, short-term domestic debt should be the dominant security issued, and that tax smoothing can be further enhanced by the government purchasing, rather than issuing, some other securities. The paper provides a rationale for why the hedging benefits of short-term domestic currency debt dominate those of the other forms of debt considered.

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1. Introduction

In the past, the literature on the optimal management of government debt has been dominated by the issue of time-inconsistency. Nominal debt creates an open invitation to create surprise inflation and erode the real value of any outstanding debt. Issuing types of debt which remove this inflationary bias dates back to Bach and Musgrave (1941). Potential solutions include indexing debt to the price level (Bach and Masgrave 1941), choosing an appropriate maturity structure (Lucas and Stokey 1983) and issuing foreign currency debt (Bohn 1991).

In recent years, a number of countries (New Zealand, Canada, the United Kingdom, Sweden, Finland, Australia and Spain) have instituted explicit inflation targeting (Svensson 1997a). Other countries, such as, Germany, Japan, and the United States, have strong implicit inflation goals. In a framework where discretionary monetary policy has an inflationary bias (e.g. Barro and Gordon 1983), Svensson (1997b) has shown that a low inflation target may reduce or even remove the inflationary bias.

With time-inconsistency issues largely resolved, it now seems appropriate to re-address the issue of public debt management. This paper uses a tax smoothing approach to find the optimal welfare maximizing public debt portfolios for nine OECD countries. The results of this paper suggest that, for eight out of the nine countries, short-term domestic debt should be the dominant security issued. When compared to actual debt portfolios, governments in these eight countries should issue more short-term domestic currency debt. Furthermore, results indicate that tax smoothing can be further enhanced by the government purchasing, rather than issuing, some other securities.

Barro (1979) was the first to develop tax smoothing theory and apply it to tax policy. He found that optimal tax policy involves the government smoothing tax rates over time. This results from his assumption that the distortionary costs of taxes are convex. These are direct costs to society, such as, deadweight losses, incentive effects, and wasted resources from tax avoidance. Minimizing these costs requires choosing a tax rate that covers the government's expenses on average. As soon as new information about the government's future ability to collect taxes becomes available, it should be used immediately to smooth tax rates over time. In contrast, transitory shocks should be borrowed against.

Bohn (1990) extended Barro's welfare approach and demonstrated that a government should use its debt portfolio to act as a hedge against any macroeconomic shocks that affect tax revenues. This approach to public finance results in a government holding a debt portfolio whose value is low when its ability to meet its repayments is low, and whose value is high when its ability to meet its repayments is high. By using its debt portfolio as a hedge, the government reduces the need to alter tax rates over time.

Although the tax smoothing approach has firm theoretical foundations[1], empirical research on debt management is still at an early stage. Empirical work includes Bohn (1990), Fowlie and Wright (1997), and Missale (1997b). Bohn (1990) found, for the period 1973-1987 in the United States, that issuing domestic debt provides greater hedging benefits, compared to indexed debt and foreign currency debt. Fowlie and Wright (1997), using a bivariate comparison between short-term domestic and foreign currency debt, found support for the issuance of foreign currency debt in New Zealand for the period 1985-1995. Finally, Missale (1997b) found a negative correlation between inflation and permanent output in Italy, but not in the United Kingdom, thus suggesting issuing nominal domestic debt provides tax smoothing benefits in Italy, but not in the United Kingdom.

This paper seeks to build on this preliminary empirical work by applying the tax smoothing approach to a sample of nine OECD countries for the post Bretton-Woods period. This enables the identification of any particular types of debt that perform consistently well across the sample of countries, in terms of their hedging abilities. In addition, Bohn’s tax smoothing approach is adapted to incorporate realistic constraints that governments face when forming their debt portfolios. When left unconstrained, Bohn’s approach typically implies governments should issue or purchase many times their GDP in particular types of debt. By putting reasonable constraints on the levels of debt that can be issued or purchased, we obtain results that can be implemented in practice, making the results more relevant to policy-makers.

[1] Missale (1997a) surveys the optimal taxation approach to public debt management.

The main results of this paper highlight the superior hedge provided by short-term debt denominated in domestic currency. The reason for short-term domestic debt's strong performance revolves around the types of shocks an economy typically faces.

When a shock to an economy is temporary, such as a monetary or fiscal demand-side shock, the government has the ability to borrow and repay its debt when the shock is over. This is what Barro (1979) described as tax smoothing. Barro proposed that deficits are varied over time to maintain constant tax rates. However, when the shock is permanent, the government loses the ability to borrow against the shock because it has affected the present value of all future tax revenues. This is the situation in which the government requires its debt portfolio to act as a hedge. These permanent shocks are typically supply-side shocks, in which prices are counter-cyclical. Therefore, in response to an adverse supply shock, output falls while prices rise. This causes the real value of short-term domestic debt to fall and act as a hedge against the adverse shock.

Foreign denominated debt and long-term debt also have this desirable inflation component in their value. However, they also have other factors influencing their value, such as exchange rates and long-term interest rates. We will argue in Section 5 that the movement of these factors may offset the benefits provided by the inflation component. In fact, the results of this paper show that lending out rather than borrowing in these forms of debt can actually enhance tax smoothing.

The rest of the paper is organised as follows. Section 2 presents the methodology of Bohn (1990) and details how it is adapted to incorporate constraints. Data used in the empirical part of the paper is introduced in Section 3. We present the optimal portfolios in Section 4 and look at various ways to measure the hedge provided by these portfolios, and by individual securities. In Section 5, using established economic theory, we seek to interpret the empirical results of the paper. Finally, Section 6 concludes and proposes some possible directions for future research.

2. Methodology

In this section, we introduce the methodology of Bohn (1990) and detail how this methodology changes as constraints are imposed. We also define the key variables of the model in detail, explaining how these variables are estimated.

2.1 The Tax Smoothing Approach

Bohn (1990) developed a framework for investigating optimal tax smoothing when a government is considering how much of K different "risky" securities to issue. By "risky", Bohn means a government security whose returns are state contingent in real terms. By this definition, government debt indexed to inflation is a "riskless" government security, denoted $k = 0$.

In his framework, there are two classes of agents: households and the government. Households are assumed to be infinitely lived, risk neutral, and expected utility maximizers. Subject to a standard inter-temporal government budget constraint, the government chooses its debt structure to maximize a representative household's utility. The chosen debt structure, the government's budget constraint and the various shocks affecting the economy determine the government's tax rate at each point in time. The key assumption is that there is a convex cost associated with the tax rate t, represented by $h(t) = \frac{h}{2} t^2$. In effect, the government's objective is to choose its debt structure to minimize the expected present value of the excess burden $h(t)$. It does this by minimizing unexpected changes in the tax rate.

The main result of this analysis is that governments should issue debt portfolios that enable taxes to be smoothed over states of nature, as well as time. Using this model, Bohn (1990), derived K optimality conditions, one for each "risky" government security k, $k = 1, \ldots, K$:

$$\sum_{i=1}^{K} Cov_t(\hat{r}_{t+1,i}, \hat{r}_{t+1,k}) d_{t,i} - w_t Cov_t(\hat{r}_{t+1,k}, \sum_{j\geq 0}^{\infty} y^{j} \hat{y}_{t+1+j}) = 0\,, \qquad \text{(I)}$$

where $d_{t,k} = \frac{p_{t,k}\, D_{t,k}}{Y_t}$ and $w_t = \left[\exp(\bar{y}) / (1 - y)\right] t_t$.

$\hat{r}_{t+1,k}$ represents the innovation in the real cost of security k. $\hat{y}_{t+1+j}$ is the innovation in the growth rate of real output j periods into the future. Both of these terms will be defined more fully in Sections 2.4 and 2.5 below. $\sum_{j\geq 0}^{\infty} y^{j} \hat{y}_{t+1+j}$ is the present value of innovations in future growth rates of real output, where y is the discount factor.

$D_{t,k}$ is the amount of government security *k* issued at time *t*, while $p_{t,k}$ is the price of security *k* and Y_t is nominal GDP. $d_{t,k}$, therefore, represents the amount of security *k* to be issued by the government as a proportion of GDP at time *t*. w_t is a weighting term where $\bar{y}$ is the average growth rate of real GDP and t_t is the average tax rate. Following Bohn, this paper uses the values of y =0.99 and t_t =0.2. This equates to a weighting term of approximately w_t =20.

In Bohn's (1990) model, the government uses tax revenues to finance government expenditures and to service government debt. $\sum_{j \geq 0}^{\infty} y^{j} \hat{y}_{t+1+j}$ captures the unexpected permanent change in the government's ability to gather tax revenues at a constant tax rate t. Bohn's original optimality condition also contained a term for the innovation in the amount of government spending relative to output. However, Bohn's research found this effect to be largely insignificant. For this reason and for the lack of quality data at a quarterly frequency, the government spending term has been disregarded in this paper.[2]

A solution to the *K* optimality conditions yields the public debt portfolio, $d_{t,k}$ (where $k = 1, \ldots, K$), with optimal tax smoothing capabilities.

2.2 The Constrained Optimum

Bohn's (1990) approach makes no allowance for the constraints a government faces when choosing its debt portfolio. His approach provides a debt portfolio with optimal tax smoothing capabilities. However, this method may produce unrealistic results for two reasons. Firstly, a solution may entail the government issuing securities far in excess of its debt requirements. For example, Bohn found for the United States that if given a choice between indexed debt and short-term domestic debt, the (net) ratio of short-term domestic debt to GDP should be 26.8; the current (gross) debt to GDP ratio in the United States is close to a half. Secondly, an unconstrained solution may see the government being a substantial net lender of a particular security.

Although governments have access to a variety of financial instruments to improve tax smoothing, there are clearly constraints on their ability to use them. A key assumption in Bohn's (1990) analysis is that asset prices and return distributions are exogenous. This implies a financial constraint that the government cannot issue or purchase such large amounts of securities that it is no longer a price taker. From a political perspective, there may be significant public opposition to issuing or purchasing securities at levels far in excess of actual debt requirements. To allow for these types of realities, Bohn's derivation of the optimality conditions is altered to incorporate constraints that:

- The government may be a net lender or borrower in any security up to an arbitrary limit. Therefore, $a_L \le d_{t,k} \le a_U, k = 0, ..., K$.
- The aggregate amount of debt that a government can issue is bounded from above by $a_{U,T}$, and below by $a_{L,T}$. Hence, $a_{L,T} \le \sum_{i=0}^{K} d_{t,i} \le a_{U,T}$.

Despite the choices of a being arbitrary, the advantage of using these constraints, which do not vary across different types of debt, is that they do not *a priori* discriminate between the $K+1$ securities a government can issue. In this sense, the constraints do not impede the objective of this paper, which is to identify government securities that help to smooth tax rates.

The optimality condition for each "risky" government security *k*, (where *k* = 1, …, *K*), now becomes[3]

$$\sum_{i=1}^{K} Cov_t(\hat{r}_{t+1,i}, \hat{r}_{t+1,k})d_{t,i} - w_t Cov_t(\hat{r}_{t+1,k}, \sum_{j \ge 0} y^{j} \hat{y}_{t+1+j}) = l_{L,k} - l_{U,k} - l_{L,0} + l_{U,0} \quad \text{(II)}$$

such that $l_{L,k} \ge 0$ and $l_{U,k} \ge 0, k = 0, ..., K$. $l_{L,k}$ is the Lagrange multiplier on the constraint that $a_L \le d_{t,k}$, and $l_{U,k}$ is the Lagrange multiplier on the constraint that $d_{t,k} \le a_{\mathrm{U}}$. $l_{L,0}$, the Lagrange multiplier on the lower constraint for indexed debt, may also be interpreted as a Lagrange multiplier on the total constraint that $\sum_{i=0}^{K} d_{t,i} \le a_{U,T}$. Likewise, $l_{U,0}$, the Lagrange multiplier on the upper bound for

[2] The IFS database does include quarterly data on government consumption. However, this series does not include spending on transfers, which are the dominant cyclical component of government spending.
[3] See Appendix 1 for a proof of this result.

indexed debt, may also be interpreted as a Lagrange multiplier on the constraint that $a_{L,T} \leq \sum_{i=0}^{K} d_{t,i}$. This last result can easily be understood. Suppose there is a constraint on total debt but not on indexed debt. Then the government can always issue or purchase the appropriate amount of indexed debt to ensure its total debt constraint is met, without affecting the K optimality conditions. More generally, it is only the constraint on indexed debt that makes a total debt constraint binding. This is why only the Lagrange multiplier on indexed debt enters equation (II).

This approach can easily be adapted to allow for other constraints. For example, because of high transaction costs, a constraint may be imposed on the amount of short-term debt relative to long-term debt issued. Due to time-inconsistency issues, a constraint may be imposed on the amount of nominal domestic currency debt relative to foreign debt issued. This adapted framework enables the tax smoothing approach to be combined with the other main considerations in a government's portfolio choice. Hence, the policy-maker can achieve the best possible tax smoothing within the constraints that other debt management issues impose.

2.3 The Real Costs of the Securities

For empirical analysis, this paper restricts attention to five of the most common types of debt that governments issue. These are:

(0) Indexed debt[4]

(1) Short-term domestic debt

(2) Short-term foreign debt

(3) Long-term domestic debt

(4) Long-term foreign debt

The real costs of these securities are as shown below. These costs represent the real cost in domestic currency that a government would incur when repaying a particular form of debt.

(0) Indexed debt $r_{t+1,0} = r$

[4] Our definition of indexed debt corresponds to a short-term security indexed to domestic inflation.

(1) Short-term domestic debt $r_{t+1,1} = i_{t,t+1} - p_{t+1}$

(2) Short-term foreign debt $r_{t+1,2} = i^*_{t,t+1} + \Delta s_{t+1} - p_{t+1}$

(3) Long-term domestic debt $r_{t+1,3} = l_t - (l_{t+1} - l_t)\mathbf{D}_{t+1} - p_{t+1}$

(4) Long-term foreign debt $r_{t+1,4} = l^*_t + \Delta s_{t+1} - (l^*_{t+1} - l^*_t)\mathbf{D}^*_{t+1} - p_{t+1}$

The real cost of indexed debt is a fixed value r, as the contract involves a promise to pay an amount contingent on inflation. The real cost of short-term domestic debt is the known nominal short-term interest rate paid, $i_{t,t+1}$, less the unknown ex-post domestic inflation rate p_{t+1}. As well as the foreign short-term interest rate $i^*_{t,t+1}$ and domestic inflation, the value of short-term foreign currency debt is affected by the percentage change in the exchange rate Δs_{t+1}, where the exchange rate s_{t+1} is defined as the natural logarithm of the domestic price of foreign currency.

The value of long-term domestic debt is influenced by the known domestic nominal long-term interest rate l_t, changes in the current long-term market interest rate, and inflation. As the long-term interest rate falls, the price of all bonds currently issued must rise to equate their yield with newly issued bonds. The currently issued bonds, therefore, become more expensive for the government to repurchase. This capital gain component is approximated by $(l_{t+1} - l_t)\mathbf{D}_{t+1}$, where $\mathbf{D}_{t+1}$ is an estimate of the modified duration as defined by Ritchken (1996). Similarly, the capital gain component of the real cost of long-term foreign currency debt is a function of the long-term foreign interest rate l^*_t, and the foreign modified duration $\mathbf{D}^*_{t+1}$.

2.4 The Innovations in the Real Costs of the Securities

We require the innovations in the real costs of the securities, in order to apply the optimality conditions (I) and (II). An innovation is the difference between an actual value and an expected value. Consequently, the innovation in the real cost of security *k*, $\hat{r}_{t+1,k}$ represents the unexpected component of the real cost, $\hat{r}_{t+1,k} = r_{t+1,k} - E_t r_{t+1,k}$. The innovations in the five securities considered are as follows:

(0) Indexed debt $\hat{r}_{t+1,0} = 0$

(1) Short-term domestic debt $\hat{r}_{t+1,1} = -(p_{t+1} - E_t p_{t+1})$

(2) Short-term foreign debt $\hat{r}_{t+1,2} = (\Delta s_{t+1} - E_t \Delta s_{t+1}) - (p_{t+1} - E_t p_{t+1})$

(3) Long-term domestic debt $\hat{r}_{t+1,3} = -(l_{t+1} - E_t l_{t+1})\mathbf{D}_{t+1} - (p_{t+1} - E_t p_{t+1})$

(4) Long-term foreign debt $\hat{r}_{t+1,4} = (\Delta s_{t+1} - E_t \Delta s_{t+1})$

$$- (l^*_{t+1} - E_t l^*_{t+1})\mathbf{D}^*_{t+1} - (p_{t+1} - E_t p_{t+1})$$

There is no unexpected change in the real cost of a domestic security indexed to inflation because by definition, its nominal return varies so that its real return is fixed. In this sense, indexed debt is the "riskless" government security (in real terms). Nominal interest rates dated t are set and known at the start of the period. Consequently, there can be no unexpected change in the known nominal interest rate component of any of the securities. Unexpected domestic inflation reduces the real value of any outstanding debt not indexed to domestic inflation, and therefore represents a negative innovation in the real costs of all securities, apart from indexed debt. Changes in the exchange rate affect the real domestic value of any foreign currency debt. For example, an unforeseen depreciation in the exchange rate unexpectedly increases the domestic value of foreign currency debt, therefore causing a positive innovation in the real cost of foreign currency debt. An unanticipated increase in the long-term interest rate, through the capital gain component, causes an unexpected fall in the price of these long-term securities, hence representing a negative innovation. By assuming modified duration is predictable, we approximate this capital gain component by the innovation in the long-term interest rate multiplied by the current modified duration.

2.5 Innovations in the Present Value of the Growth Rate of Output

From a tax smoothing point of view, the most desirable characteristic of a government liability is its ability to act as a hedge against any permanent unexpected changes in the government's ability to gather tax revenues. If the value of a country's debt portfolio falls when the economy receives a permanent unexpected adverse shock, there will be a reduced need on the part of the policymaker to increase tax rates. The innovation in the growth rate of real output, j periods into the future, is given by $\hat{y}_{t+1+j} = E_{t+1} y_{t+1+j} - E_t y_{t+1+j}$. It is the difference in expectations from time t to time $t+1$, of the real output growth in period $t+1+j$. It is essentially a measure of the change in the expected value of real output growth in period $t+1+j$, given the newly available information at time $t+1$.

The innovation in the present value of future real output growth is given by $\sum_{j\geq 0}^{\infty} y^{j} \hat{y}_{t+1+j}$. A time series for the present value of innovations in the future growth rate of real output is estimated as follows. Starting at the first time period, forecasts are formed recursively at time t of real output growth in periods $t+1,\ t+2,\ \ldots,\ t+n$ using the forecasting techniques detailed in Section 2.6 below. This gives values of the expected growth of real output in periods $t+1,\ t+2,\ \ldots,\ t+n$, given the available information at time t. Forecasts of real output growth in periods $t+1,\ t+2,\ \ldots,\ t+n$, are then formed recursively at time $t+1$. These give values of the expected growth of real output in periods $t+1,\ t+2,\ \ldots,\ t+n$, given the available information at time $t+1$. The difference in forecasts from time t to $t+1$ is then taken, giving values of $\hat{y}_{t+1+j} = E_{t+1} y_{t+1+j} - E_t y_{t+1+j}$, for all values of j. This procedure is repeated at every time period to obtain a time series for the present value of innovations in the future growth rate of real output.[5]

2.6 Calculating Expectations

To calculate innovations in the real costs of the different securities considered, expectations need to be formed at time t of the inflation rate p_{t+1}, the percentage change in the exchange rate Δs_{t+1}, the long-term domestic interest rate l_{t+1}, and the long-term foreign interest rate l^{*}_{t+1}. Furthermore, to calculate innovations in the future growth rate of real output $\hat{y}_{t+1+j}$, expectations need to be formed at time t and time $t+1$ of the growth rate of real output y_{t+1+j}, for $j=0$ through to $j=50$.

[5] It was found in practise that $\hat{y}_{t+1+j}$ converged to zero for values of j greater than 50. For this reason, the innovation in the present value of future real output growth is calculated using innovations in growth rates up to $j=50$ quarters into the future (12.5 years), with an adjustment made for this approximation.

Vector Autoregressions (VARs) are used to calculate expectations in the required variables. This method was chosen because of its suitability for computing projections in a multivariate framework.

The initial specification of the forecasting equations was that of a standard VAR. Expectations are formed for each variable $(p_{t+1}, \Delta s_{t+1}, l_{t+1}, l^*_{t+1}, y_{t+1})$ by regressing that variable on a constant and four lags of the variable, together with four lags of all other variables. Four lags were used in accordance with Bohn (1990).[6] We know, however, when forecasting the exchange rate, a simple full VAR model of this kind generally performs worse than a simple random walk (Messe and Rogoff 1983). We have good reason to believe many of the other variables in the model are also redundant so including them is only likely to add noise to our forecasts (for example, the exchange rate between Australia and Germany is unlikely to have predictive power for the Japanese long-term interest rate).

To improve the forecasting ability of our model from that of a standard VAR, we searched over all lag lengths, with up to four lags for each variable to find the specification of each forecasting equation that minimized the Schwarz Bayesian Information Criterion (SBIC). Minimizing the SBIC provides a specification of the forecasting equations with optimal forecasting properties (Gweke and Messe 1981). No theory is imposed on the specification of the forecasting equations; explanatory variables are merely chosen for their ability to minimize the SBIC. This procedure is repeated separately for each of the nine countries in the sample. Appendix 2 contains the resulting specifications.[7]

3. Data

Data on exchange rates, long-term interest rates, nominal GDP, and real GDP were collected from the IFS database. Countries making up the sample are Australia, Canada, Germany, Italy, Japan, the Netherlands, Spain, the United Kingdom, and the United States. These countries were chosen on the basis of the availability of the required data.

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[6] Three foreign countries are used to measure exchange rates and foreign interest rates. Thus, including a constant, there are at most 37 explanatory variables for each forecasting equation, which is estimated in the VAR.

The exchange rate data collected is the exchange rate against the United States dollar. Other exchange rates are calculated by triangular arbitrage. Long-term interest rates are the long-term government bonds yield series available from the IFS. The maturities of these government bonds vary between 10 years for the United States and 20 years for the United Kingdom. Inflation is estimated by taking log differences of the GDP deflator. Growth rates are estimated by taking log differences of real GDP.

All data is seasonally adjusted. All changes are quarterly and long-term interest rates are converted into quarterly returns. The sample is the post-Bretton Woods period of 1975:1 - 1996:4, where most countries operated floating exchange rate regimes. Due to their dominance of world capital markets, the German mark, Japanese yen, and the United States dollar are used as foreign currencies that debt can be denominated in. The German mark is not considered a foreign currency for Italy, the Netherlands or Spain, as the available data is affected by the period of fixed exchange rates as part of the European Monetary System.[8]

4. Results

In this section, we present the optimal public debt portfolios, both unconstrained and constrained, found by applying our methodology to actual data. We also seek to measure the improvement in tax smoothing provided by these optimal portfolios. Finally, we attempt to isolate the contribution to tax smoothing made by individual securities, thereby identifying the securities that provide the greatest tax smoothing benefits.

4.1 Optimal Unconstrained Portfolios

The optimal tax smoothing portfolios are at first found in the case where there are no constraints.[9] The optimal portfolios are shown in Table 1.

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[7] A random walk was found to be the best predictive model for the exchange rate in 17 out of 21 of the forecasting equations.

[8] The United Kingdom was also briefly a member of the EMS. As the membership was only from 1990 to 1992, Germany is considered a foreign currency for the United Kingdom.

[9] Results were also found using standard VARs with four lags on all variables. The results are contained in Appendix 3. Unfortunately, none of the results were statistically significant (t-statistics

Table 1. Optimal Unconstrained Public Debt Portfolios

	d_1	d_2^{US}	d_2^{GER}	d_2^{JAP}	d_3	d_4^{US}	d_4^{GER}	d_4^{JAP}
AUS	7.27 ***	0.06 **	0.02	0.12*	0.06 **	-0.38***	0.07* **	-0.10***
CAN	2.08 ***	0.27***	-0.05	0.21	0.56***	-1.19***	-0.12	0.13
GER	12.06 ***	0.03		0.06**	-0.26 **	0.09 **		0.02
ITA	-1.83 ***	-0.04		0.57***	0.11***	-0.09***		-0.43***
JAP	0.72 ***	0.09***	0.11*		-0.23***	-0.13***	-0.10*	
NET	5.14 ***	0.05		0.16***	-0.41***	-0.15		-0.16***
SPA	0.04	0.62***		0.35***	0.19***	-0.55***		-0.34***
UK	7.23 ***	-0.12 *	0.02	0.21*	-0.02	0.06*	0.03	-0.24 *
USA	4.53 ***		-0.17*	-0.16***	-0.58***		0.02	0.32***

Note: d_i denotes the amount of government security *i*, included in the government's debt portfolio as a proportion of annual GDP, where i = 0, 1, 2, 3, 4. 0 is indexed debt, 1 is short-term domestic debt, 2 is short-term foreign debt, 3 is long-term domestic debt, and 4 is long-term foreign debt. A superscript denotes the foreign currency that the foreign debt is denominated in. Standard Monte Carlo methods are used to find if results are statistically different from zero. *** denotes significance at a 1% level, ** denotes significance at a 5% level and * denotes significance at a 10% level.

The first notable feature of these results is the extent to which short-term domestic debt (d_1) dominates the optimal portfolios. For Australia, Canada, Germany, the Netherlands, the United Kingdom, and the United States, the optimal amount of short-term debt to issue exceeds twice annual GDP. Short-term foreign debt, especially short-term Japanese currency debt, also displays some consistency, although the optimal amounts of these debt are generally much smaller than short-term domestic debt. Short-term US debt appears quite important for Canada and Spain.

Long-term domestic debt (d_3) displays mixed results. The countries in the sample are split, as to whom should borrow or lend, in this form of debt. Results are more clear-cut for long-term foreign currency debt, suggesting that for most countries, it is optimal to buy bonds rather than sell bonds in US dollars, in German marks, and in Japanese yen. This result is especially apparent for long-term United States currency debt. To enhance domestic tax smoothing, six out of a possible eight counties should lend out, rather than borrow, this form of debt.

Indexed debt (d_0) cannot contribute to the hedge provided by the optimal unconstrained portfolio. The optimal amount of indexed debt to issue is, therefore, undefined. The only reason to issue or purchase indexed debt would be to meet some constraint on total debt.

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varied between -0.18 and 0.21), so we are not even close to rejecting that these results occurred by

4.2 Optimal Constrained Portfolios

The portfolios in Table 1 involve taking short or long positions, well in excess of the country's GDP. For example, Germany would issue more than twelve times its GDP in short-term domestic debt, while Canada would purchase an amount in excess of its GDP in long-term debt denominated in United States dollars. In this section, we explore restrictions on the government's ability to issue and purchase securities.

4.2.1 No Lending Allowed

Governments are generally constrained to use their debt portfolios to satisfy their debt requirements, rather than purely for tax smoothing purposes. To allow for this fact, two constraints are imposed. Firstly, a government's total debt cannot exceed annual GDP. This is an arbitrary constraint but represents the fact that there exists some upper limit on a government's ability to borrow. The second constraint imposed is that a government cannot be a net lender of any security. This follows from the assumption that governments use their debt portfolios to satisfy their debt requirements and are, therefore, debtors rather than creditors, in financial markets. Table 2 details results of the tax smoothing approach in this constrained environment.

Table 2. Optimal Constrained Public Debt Portfolios (1)

	d_1	d_2^{US}	d_2^{GER}	d_2^{JAP}	d_3	d_4^{US}	d_4^{GER}	d_4^{JAP}	d_0
Australia	0.96	0.00	0.02	0.00	0.02	0.00	0.00	0.00	0.00
Canada	0.92	0.00	0.00	0.08	0.00	0.00	0.00	0.00	0.00
Germany	0.75	0.25		0.00	0.00	0.00		0.00	0.00
Italy	0.00	0.00		0.00	0.03	0.00		0.00	a
Japan	0.33	0.02	0.01		0.00	0.00	0.00		a
Netherlands	1.00	0.00		0.00	0.00	0.00		0.00	0.00
Spain	0.45	0.17		0.00	0.02	0.00		0.00	a
UK	0.97	0.01	0.02	0.00	0.00	0.00	0.00	0.00	0.00
USA	0.98		0.00	0.00	0.00		0.00	0.02	0.00

Note: Significance testing is not performed as results are constrained. (a) The optimal amount of indexed debt issued can vary between 0 and that amount which makes the constraint on total debt binding. See Table 1 for other definitions.

chance.

The above results represent realistic portfolios. For Australia, Canada, the Netherlands, the United Kingdom, and the United States optimal tax smoothing requires that short-term domestic debt makes up over 90% of the public debt portfolio. In the case of Germany, short-term domestic debt accounts for 75% of the optimal portfolio. Although the optimal amount of short-term domestic debt is much smaller in Japan and Spain, it is still the dominant security in their portfolios. Thus, for eight out of nine countries, short-term domestic debt should be the dominant security issued.

No other types of debt consistently feature in these constrained portfolios. However, these portfolios do show that other forms of debt should not be disregarded. Netherlands is the only country for which it is optimal to issue 100% of its portfolio in short-term domestic debt. For all other countries, it is optimal to issue small amounts of other forms of debt, such as short-term foreign debt or long-term domestic debt. In effect, the results illustrate the importance of not just choosing a single form of debt. Instead, a portfolio that incorporates various forms of debt, which together provide the best hedge, should be chosen.

4.2.2 Limited Lending Allowed

Many governments hold assets, domestically or in foreign countries, such that the net amount of certain forms of debt is actually negative. This suggests that there exists some scope for governments to participate as net lenders and perhaps, move closer to the optimal tax smoothing portfolios described in Table 1. In this section, we relax the constraint that governments cannot lend and instead impose a constraint that governments can lend out at most 10% of annual GDP in each form of security. The relevant constraint is then $-0.10 \leq d_{t,k}$, $\forall k$, $k = 0, \ldots, K$. Results are detailed in Table 3 below.

Now that lending is allowed, all countries lend out some form of long-term foreign currency debt. Six out of a possible nine countries will also lend out their full requirement of indexed debt. These two actions allow every country, except Italy and Spain, to increase their positions in short-term domestic debt, hence gaining from its hedging properties. Being able to lend out long-term foreign currency debt introduces a type of tax smoothing leverage. Tax smoothing is enhanced by purchasing foreign currency debt, while at the same time, it can be enhanced further as the government can now issue more short-term domestic debt without breaching the constraint on total debt.

Table 3. Optimal Constrained Public Debt Portfolios (2)

	d_1	d_2^{US}	d_2^{GER}	d_2^{JAP}	d_3	d_4^{US}	d_4^{GER}	d_4^{JAP}	d_0
Australia	1.11	-0.10	0.06	0.17	0.06	-0.10	0.00	-0.10	-0.10
Canada	1.41	-0.10	-0.10	0.00	-0.10	-0.10	-0.10	0.19	-0.10
Germany	0.83	0.28		0.10	0.04	-0.05		-0.10	-0.10
Italy	-0.10	0.00		0.13	0.04	-0.10		-0.10	b
Japan	0.53	0.08	0.11		-0.10	-0.10	-0.10		b
Netherlands	1.23	0.05		0.12	-0.10	-0.10		-0.10	-0.10
Spain	0.44	0.31		-0.05	0.03	-0.10		-0.10	b
UK	1.00	0.12	0.21	-0.10	0.06	-0.10	-0.10	0.01	-0.10
USA	1.26		-0.07	-0.09	-0.10		-0.10	0.20	-0.10

Note: Significance testing is not performed as results are constrained. (b) The optimal amount of indexed debt issued can vary between -0.10 and that amount which makes the constraint on total debt binding. See Table 1 for other definitions.

Another notable feature of this section is that indexed debt now enters the optimal constrained portfolios, when it previously had not. The reason for this, is as follows. The real cost of indexed debt is fixed and hence it cannot act as a hedge against any macroeconomic shocks. Therefore, indexed debt has a neutral effect on the hedging abilities of a portfolio. However, in the case of Australia, Canada, Germany, the Netherlands, the United Kingdom, and the United States, there now exists a binding constraint on total debt. It becomes optimal to lend out indexed debt as it allows the government to borrow more of the forms of debt with hedging properties without violating the constraint on total debt.

4.3 Tax Smoothing and Welfare Gains

The actual composition of public debt, as a proportion of total debt, and the debt to GDP ratios of the countries in our sample are shown in Table 4. This table is based on data for 1989 in DeBroeck (1997) and represents gross amounts of debt. Net amounts of debt would provide a more appropriate comparison for our analysis. However, as Missale (1994) pointed out, data on net amounts of debt are difficult to compute and unreliable as they depend heavily on definitions of public assets and subsequent valuation techniques. Consequently, we will use these gross values as a proxy for net public debt in what follows. Furthermore, we assume that all foreign currency debt is long-term and spread evenly over the applicable foreign currencies.

Table 4. Actual Debt Composition in 1989

	Short term Domestic Debt (1)	*Long Term Domestic Debt (3)*	*Foreign Currency Debt (4)*	*Debt to GDP ratio*
Australia	0.24	0.59	0.17	0.16
Canada	0.40	0.58	0.02	0.44
Germany	0.03	0.97	0.00	0.20
Italy	0.72	0.24	0.04	0.93
Japan	0.00	1.00	0.00	0.30
Netherlands	0.01	0.99	0.00	0.61
Spain	0.64	0.33	0.03	0.34
United Kingdom	0.04	0.90	0.06	0.32
United States	0.19	0.81	0.00	0.47

Note: Debt composition excludes government debt held by the Central Bank.

As with the results from optimal tax smoothing, foreign currency debt generally makes up only a small proportion of total government debt. However, unlike optimal tax smoothing, the actual public debt portfolios are made up predominantly of long-term domestic debt. From a tax smoothing point of view, this suggests that there are significant welfare gains to be made from the reallocation of public debt. More specifically, countries can enhance tax smoothing and, as a result, improve the welfare of a representative household by issuing less long-term domestic debt and more short-term domestic debt; this is true for eight out of the nine countries, as can be seen by comparing Table 4 with Tables 2 or 3.

We use the variance of innovations in the tax rate as a measure of tax smoothing under the various debt portfolios considered. Bohn's (1990) approximate solution for the $t+1$ period innovation in the tax rate is used to calculate the required variances (equation [9], p.1223). This approximation is the same as that used in deriving the K optimality conditions (I) and (II).[10]

$$\begin{aligned} \hat{t}_{t+1} &= t_{t+1} - E_t t_{t+1} \\ &= (1-y)\cdot \exp(-\bar{y}) \times \left[\sum_{i=1}^{K} \hat{r}_{t+1,i} d_{t+1,i} \right] - t \sum_{j \geq 0}^{\infty} y^{j} \hat{y}_{t+1+j} . \end{aligned} \qquad \text{(III)}$$

To estimate the improvement in tax smoothing provided by the optimal portfolios above, we use the percentage reduction in the variance of innovations in the tax rate achieved by moving from the actual portfolios (Table 4) to the optimal

[10] As before, the innovation in government spending term has been omitted.

portfolios. To show how the optimal portfolios compare to a neutral position, we also calculate the reduction in variance caused by moving from a portfolio of indexed debt to the optimal portfolios. Finally, we calculate the percentage reduction in variance caused by moving from the actual portfolio to a portfolio of indexed debt. This gives an indication of the hedging properties that the current actual portfolios exhibit. Results are shown in Table 5.

Table 5. Possible Improvements in Tax Smoothing

	AO	*AC1*	*AC2*	*IO*	*IC1*	*IC2*	*AI*
Australia	59	12	18	59	14	19	-2
Canada	25	5	12	22	2	9	4
Germany	50	13	15	50	14	15	0
Italy	15	7	10	9	0	3	7
Japan	18	12	17	7	1	6	12
Netherlands	53	19	24	53	18	23	1
Spain	13	2	7	14	3	8	-1
UK	78	20	24	79	21	25	-1
US	43	22	30	31	6	15	17

Note: The improvement in tax smoothing is measured as the percentage reduction in the variance of innovations in the tax rate. AO is from actual to optimal; AC1 is from actual to constrained (1); AC2 is from actual to constrained (2); IO is from indexed to optimal; IC1 is from indexed to constrained (1); IC2 is from indexed to constrained (2); and AI is from actual to indexed.

These results highlight the economic benefits possible from issuing the optimal tax smoothing public debt portfolios, as opposed to the current actual portfolios. For Australia, Germany, the Netherlands, the United Kingdom, and the United States this reallocation of public debt would reduce the variance of innovations in the tax rate by over 40%. At 78%, the United Kingdom takes a large step towards completely eliminating unexpected changes in the tax rate. Similar benefits to those above accrue when moving from indexed debt to the optimal portfolio.

Significant reductions in variance occur even when the portfolios are constrained. In the case where lending is limited to 10% of annual GDP in each security, variance can still be reduced by over 20% in the Netherlands, the United Kingdom and the United States, and by over 10% in all but one country. Another notable feature of these results is that for Canada, Italy, Japan, the Netherlands, and the United States tax smoothing can be improved by moving from their current portfolios to portfolios of indexed debt. This is a particularly severe indictment on the tax smoothing properties of the current portfolios, as we know that indexed debt provides no hedging benefits.

4.4 Investigating the Source of the Hedging Abilities

It has been shown so far that short-term domestic debt dominates the optimal portfolios of most countries in the sample. Furthermore, it has been shown that there are non-trivial gains to be made by moving to these optimal positions and benefiting from the increased hedge they provide. In this section, we investigate the source of these hedging benefits using two alternative approaches.

4.4.1 The Cost of Moving Away from the Optimal Portfolio

The first approach investigates how the variance of innovations in the tax rate changes as we move away from the optimal portfolios detailed in Table 1. To do this, we find the percentage increase in the variance of innovations in the tax rate caused by moving from the optimal portfolio to a portfolio identical to the optimal portfolio, but where one of the types of debt is instead set to zero. In effect, we are measuring the cost, in terms of increased unexpected variation in tax rates, incurred by not being able to borrow or lend that security. This procedure is repeated for each security to gain an appreciation of each security's contribution to tax smoothing. The results are shown in Table 6.

Table 6. The Cost of a Shift from the Optimum

	1	*2 (US)*	*2 (GER)*	*2 (JAP)*	*3*	*4 (US)*	*4 (GER)*	*4 (JAP)*	*0*
Australia	138	0	0	3	2	26	2	3	0
Canada	3	1	0	7	12	51	2	3	0
Germany	93	0		0	2	1		0	0
Italy	5	0		30	1	1		29	0
Japan	2	2	2		3	4	2		0
Netherlands	127	0		4	5	5		6	0
Spain	0	24		7	1	22		10	0
UK	395	5	0	16	1	1	0	33	0
USA	38		5	4	31		0	26	0

Note: The cost of a shift from the optimum is measured as the percentage increase in the variance of innovations in the tax rate. The countries that foreign debt is denominated in are shown in brackets. See Table 1 for other definitions.

For five out of the nine countries, the primary source of tax smoothing, at the optimum, is from the issuance of short-term domestic debt. By not allowing the issue of short-term domestic debt, the variance of innovations in the tax rate approximately doubles, compared to the optimum, for Australia, Germany and the Netherlands. For the United Kingdom the variance would increase nearly five-fold.

The large costs of not issuing short-term domestic debt compared to the optimum for Australia, Germany, the Netherlands, the United Kingdom, and the United States, reflects both that at the optimum these countries should issue a lot of short-term domestic debt, and that in practise they issue very little short-term domestic debt. Table 5 reflects these results: these countries achieve the five largest gains from moving from their actual portfolios to their unconstrained optimal portfolios, as well as to their no-lending optimal portfolios. Table 6 also highlights the cost on tax smoothing of not being able to purchase securities. From an unconstrained perspective, it is optimal for the United States to lend out 58% of its GDP in long-term domestic debt. By not doing this, the variance of innovations in the tax rate increases by 31%. In reality, the United States actually borrows around 40% of its GDP in long-term domestic debt.

4.4.2 The Benefit of Issuing Additional Debt

Another way to isolate the hedging ability of individual securities would be to determine how much tax smoothing improves as a result of issuing an additional amount of an individual security. To do this, we calculate the reduction in the variance of innovations in the tax rate achievable by issuing 100% of GDP in a single form of debt instrument, starting from the position of holding a portfolio of indexed debt.[11] Moving from the position of a portfolio of indexed debt provides a neutral starting point for the analysis. Results are shown in Table 7 below.

Table 7 shows that if governments choose to issue 100% of GDP in a single form of security and smooth taxes, that security should either be short-term domestic debt or indexed debt. Short-term domestic debt is the only form of security, when issued in this amount, that can provide tax smoothing benefits. Even in Italy and Spain, where there is in fact a cost from moving away from indexed debt, short-term domestic debt provides the best hedge of any non-indexed security. Once again, these results provide evidence that issuing large amounts of long-term debt, both domestic and foreign, may impose high costs in terms of tax smoothing. This is evident from the

[11] We know from Table 4 that Germany, Japan, and the Netherlands each issue almost exclusively one form of debt instrument. This suggests that these types of large one off choices of debt do occur.

large increases in the variance of innovations in the tax rate caused by issuing 100% of GDP in these securities.

Table 7. The Tax Smoothing Benefit of Individual Securities

	1	*2 (US)*	*2 (GER)*	*2 (JAP)*	*3*	*4 (US)*	*4 (GER)*	*4 (JAP)*	*0*
Australia	14	-40	-84	-77	-20	-84	-104	-124	0
Canada	1	-18	-107	-76	-38	-60	-136	-127	0
Germany	7	-43		-38	-12	-48		-86	0
Italy	-6	-115		-90	-52	-129		-172	0
Japan	-1	-196	-132		-80	-213	-174		0
Netherlands	18	-94		-55	-8	-102		-113	0
Spain	1	-32		-59	-25	-72		-104	0
UK	21	-48	-29	-70	-41	-76	-69	-113	0
USA	7		-125	-106	-97		-165	-163	0

Note: The tax smoothing benefit of each security is measured as the reduction in the variance of innovations in the tax rate caused by issuing 100% of GDP in that security (compared to a portfolio of indexed debt). See Table 6 for other definitions.

5. An Interpretation of our Findings

Some significant trends have emerged from the empirical results presented. Firstly, issuing short-term domestic debt seems to provide the government with the best hedge against future income. Secondly, there seems to be a role in purchasing other forms of securities, such as long-term foreign currency debt, to improve tax smoothing. In this section, we ask what economic theory would imply for the tax smoothing merits of the securities we have considered. The key relationships that need to be determined are the co-movements between shocks to permanent output growth and the components of the real costs of the securities we consider, those being: the inflation rate, the exchange rate, and both domestic and foreign long-term interest rates.

Aggregate demand side shocks such as monetary and fiscal shocks are generally thought to be temporary. Temporary shocks will not significantly affect the present value of future government revenues, and therefore these shocks may be borrowed against. Supply side shocks such as technology and productivity shocks are generally thought to be permanent in nature. A technology shock will therefore have a significant effect on the present value of future government revenues. This is the situation in which the government requires its debt portfolio to act as a hedge. Typically, prices are counter-cyclical in response to aggregate supply side shocks (Den Haan, 1996). This means that when there is a permanent adverse supply side shock, prices will rise and the real domestic value of all outstanding government debt, not indexed to inflation, will fall. Therefore, the inflation component of the real cost of government debt provides valuable hedging properties.

All of the securities considered, not indexed to inflation, have this desirable inflation component in there value. However, short-term domestic debt is the only security whose unexpected real cost is solely dependent on unexpected inflation. The theoretical explanation in favor of the inflation component is therefore, a direct case in favor of short-term domestic debt.

A classic example of the hedging qualities of short-term domestic would be the behavior of inflation and GDP growth that typically occurs during wars.[12] During wars, military spending expands rapidly, while at the same time, tax revenues fall as a large portion of the workforce is lost. Both of these occurrences serve to put considerable strain on the government's budget constraint. However, wars are also typically accompanied by periods of significant inflation. By holding a portfolio of short-term domestic debt, the real value of the government's outstanding liabilities falls, due to the increase in the price level during a time when government expenses are high and revenues are low.

Changes in the exchange rate affect the real cost of all foreign currency debt. The classic theory of Purchasing Power Parity (PPP) suggests that the exchange rate moves one-for-one to offset changes in the domestic price level. The extent to which PPP holds in the long run, the hedging benefits from the domestic inflation component of foreign currency debt will be undone by movements in the exchange rate.[13] In addition, productivity effects (Balassa 1964 and Samuelson 1964) suggest that the domestic currency will appreciate more than expected when productivity growth is higher than expected. Since this occurs when permanent output is higher, it may be optimal for governments to actually purchase, not issue, foreign currency debt. By doing this, they hold an asset whose value is high, through a depreciation in the exchange rate when income is low. Holding this asset would reduce the need to alter tax rates as the asset's value acts as a hedge against future tax revenues.

[12] Another example would be an oil shock, in which higher prices are associated with lower output.
[13] To the extent the inflation is world-wide, the hedging benefits from the inflation component will remain.

The value of long-term debt is also affected by changes in the long-term interest rate. In the short run, standard economic theory suggests that interest rates are determined predominantly by monetary factors. High interest rates are associated with tight monetary policy, hence, low output growth. However, according to this theory, changes in monetary policy only have temporary effects on output growth and therefore can be mostly borrowed against. Conversely, according to the classical view of macroeconomics, in the long run interest rates are determined by the marginal product of capital. Therefore, an adverse supply side shock, which reduces the marginal product of capital, will result in a reduction of the interest rate. This, in turn, increases the price of long-term government debt, through the capital gain component. If this capital gain component dominates the inflation component discussed above, it suggests that in order to improve tax smoothing, long-term domestic debt should be purchased, not issued. Furthermore, to the extent long-term interest rates available in different currencies are positively correlated, this conclusion is also likely to hold for long-term foreign currency debt.

In summary, the interpretation presented suggests the inflation component of the real cost of debt provides a consistent hedge against permanent income. However, the co-movement of exchange rates and interest rates with productivity suggests that it may be optimal for foreign currency debt and long-term debt to be held, not issued, by governments. This leaves short-term domestic currency debt as the only form of public debt that provides a consistent hedge against future income. This hedge is provided by inflation’s co-movement with permanent shocks to the macroeconomy. The theoretical interpretation presented above suggests that our results may, in fact, be more general than just the nine countries studied or the time period examined.

6. Concluding Remarks

Bohn (1990) developed a tax-smoothing methodology to find the optimal government debt portfolio for the United States. This approach may, however, produce results that are not possible to implement. This paper adapts Bohn’s methodology and imposes realistic constraints on public debt management. More importantly, this constrained approach is applied to nine OECD countries.

Although the results are not completely conclusive, the common theme that arises from the results is that the issuance of short-term domestic currency debt provides the best hedge against future income. Furthermore, purchasing, instead of issuing, long-term foreign currency debt and indexed debt can provide a type of tax smoothing leverage. By lending out these forms of debt, more of the forms of debt with hedging properties can be issued without breaching a possible constraint on total debt.

The poor performance of long-term debt, when using a hedging criterion, calls into question the current debt choice of the countries in the sample, which predominantly issue this form of debt. This is further highlighted, as the benefits of moving from the current actual portfolios to the optimal ones, are found to be economically significant. Given the potential benefits at stake, further research using alternative forecasting approaches seems warranted. One such approach might be to exploit potential cointegration relationships between the forecasted variables. Another approach would be to allow time-varying parameters by way of Kalman Filter estimation.

This paper could be best described as an empirical paper with a theoretical explanation. Bohn's (1990) methodology is adapted and applied to a sample of countries to find the policy implications of the tax smoothing approach. Another way to deduce the implications of the tax smoothing approach would be to build a small macroeconomic model. Aggregate demand and supply shocks could be modelled and the effects of these shocks on the exchange rate, long-term interest rates, and inflation found. The optimal debt portfolios could be constructed based on the co-variation between these simulated series. A theoretical model of this form could perhaps further explain the empirical findings of this paper.

Appendix 1. Proof for the Constrained Optimum

We start by repeating some results from Bohn (1990). We use his formulation for the household's objective function, equation (5), p.1220,

$$U_t = E_t \sum_{j \geq 0}^{\infty} r^j \left\{ Y_{t+j} \left[1 - h(t_{t+j}) \right] \right\} \qquad \text{(IV)}$$

where r is a discount factor. We also use his formulation of the government's budget constraint, equation (4), p.1220,

$$t_t Y_t = G_t + \sum_{i=0}^{K} \left(P_{t,i} + f_{t,i} \right) D_{t-1,i} - \sum_{i=0}^{K} P_{t,i} D_{t,i} \qquad \text{(V)}$$

where $f_{t,i}$ is the interest payment on government security i at time t. Following Bohn, we assume a quadratic excess loss function

$$h(t_t) = \frac{h}{2} t_t^2 \qquad \text{(VI)}$$

We substitute (V) and (VI) into (IV), to get

$$U_t = E_t \sum_{j \geq 0}^{\infty} r^j \left\{ Y_{t+j} \left(1 - \frac{h}{2} \left[\frac{G_{t+j}}{Y_{t+j}} + \sum_{i=0}^{K} (P_{t+j,i} + f_{t+j,i}) \frac{D_{t+j-1,i}}{Y_{t+j}} - \sum_{i=0}^{K} \frac{P_{t+j,i} D_{t+j,i}}{Y_{t+j}} \right]^2 \right) \right\} \qquad \text{(VII)}$$

The government chooses debt structure to maximize (VII), subject to the following constraints:

$a_{L,i} Y_t \leq P_{t,i} D_{t,i},\ i = 0, \ldots, K.$ (lower bound on individual debt)

$P_{t,i} D_{t,i} \leq a_{U,i} Y_t,\ i = 0, \ldots, K.$ (upper bound on individual debt)

$a_{\mathrm{L,T}} Y_t \leq \sum_{i=0}^{K} P_{t,i} D_{t,i}$ (lower bound on total debt)

$\sum_{i=0}^{K} P_{t,i} D_{t,i} \leq a_{\mathrm{U,T}} Y_t$ (upper bound on total debt)

The appropriate Lagrangian is

$$L = U_t + \sum_{i=0}^{K} l_{L,i} \left(P_{t,i} D_{t,i} - a_{L,i} Y_t \right) + \sum_{i=0}^{K} l_{U,i} \left(a_{U,i} Y_t - P_{t,i} D_{t,i} \right) + l_{U,T} \left(a_{U,T} Y_t - \sum_{i=0}^{K} P_{t,i} D_{t,i} \right) + l_{L,T} \left(\sum_{i=0}^{K} P_{t,i} D_{t,i} - a_{L,T} Y_t \right)$$

The first order conditions, one for each "risky" government security k, $k = 1, \ldots, K$, become

$$\frac{\P L}{\P D_{t,k}} = h Y_t \left(\frac{G_t}{Y_t} + \sum_{i=0}^{K} (P_{t,i} + f_{t,i}) \frac{D_{t-1,i}}{Y_t} - \sum_{i=0}^{K} \frac{P_{t,i} D_{t,i}}{Y_t} \right) \frac{P_{t,k}}{Y_t}$$
$$- E_t \left\{ r h Y_{t+1} \left(\frac{G_{t+1}}{Y_{t+1}} + \sum_{i=0}^{K} (P_{t+1,i} + f_{t+1,i}) \frac{D_{t,i}}{Y_{t+1}} - \sum_{i=0}^{K} \frac{P_{t+1,i} D_{t+1,i}}{Y_{t+1}} \right) \frac{P_{t+1,k} + f_{t+1,k}}{Y_{t+1}} \right\}$$
$$+ l_{L,k} P_{t,k} - l_{U,k} P_{t,k} - l_{U,T} P_{t,k} + l_{L,T} P_{t,k} = 0$$

Writing this first order conditions in terms of t_t, using (V), yields

$$ht_t P_{t,k} - rhE_t\left\{t_{t+1}(P_{t+1,k} + f_{t+1,k})\right\} + l_{L,k}P_{t,k} - l_{U,k}P_{t,k} - l_{U,T}P_{t,k} + l_{L,T}P_{t,k} = 0 \quad \text{(VIII)}$$

Dividing (VIII) through by $rhP_{t,k}$ gives

$$\frac{t_t}{r} - E_t\left\{t_{t+1}(1 + r_{t+1,k})\right\} + \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{U,T}}{rh} + \frac{l_{L,T}}{rh} = 0 \quad \text{(IX)}$$

We use Bohn's condition for individual optimization; equation (3), p.1219, shown below

$$E_t\left(1 + r_{t+1,k}\right) = \frac{1}{r} \text{ for any k.} \quad \text{(X)}$$

Substituting (X) into (IX) yields,

$$E_t\left\{(t_{t+1} - t_t)(1 + r_{t+1,k})\right\} = \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{U,T}}{rh} + \frac{l_{L,T}}{rh} \quad \text{(XI)}$$

which can be re-written as

$$E_t(t_{t+1} - t_t)E_t(1 + r_{t+1,k}) + Cov_t(t_{t+1} - t_t,\ 1 + r_{t+1,k}) = \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{U,T}}{rh} + \frac{l_{L,T}}{rh} \quad \text{(XII)}$$

Using (X), (XII) can be further simplified to

$$\frac{E_t(t_{t+1} - t_t)}{r} + Cov_t(t_{t+1} - t_t,\ 1 + r_{t+1,k}) = \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{U,T}}{rh} + \frac{l_{L,T}}{rh} \quad \text{(XIII)}$$

Evaluating (XIII) for indexed debt ($k = 0$), the covariance term becomes zero and (XIII) becomes

$$\frac{E_t(t_{t+1} - t_t)}{r} = \frac{l_{L,0}}{rh} - \frac{l_{U,0}}{rh} - \frac{l_{U,T}}{rh} + \frac{l_{L,T}}{rh} \quad \text{(XIV)}$$

Substituting (XIV) into (XIII) and canceling common terms yields

$$Cov_t(t_{t+1} - t_t,\ 1 + r_{t+1,k}) = \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{L,0}}{rh} + \frac{l_{U,0}}{rh}$$

which is equivalent to

$$Cov_t(\hat{t}_{t+1},\ \hat{r}_{t+1,k}) = \frac{l_{L,k}}{rh} - \frac{l_{U,k}}{rh} - \frac{l_{L,0}}{rh} + \frac{l_{U,0}}{rh}$$

Note that the only restriction on the Lagrange multipliers is that they are non-negative, so we can set rh to be one without loss of generality. This gives:

$$Cov_t(\hat{t}_{t+1}, \hat{r}_{t+1,k}) = l_{L,k} - l_{U,k} - l_{L,0} + l_{U,0}$$

By substituting in Bohn's approximation for the innovation in the tax rate (equation (III)), we obtain the first order condition for the constrained optimum, equation (II), as required. □

Appendix 2. Specification of the Reduced VARs

For each country, we regress each dependent variable on a constant and a set of other regressors. We list this set beside each dependent variable below. Where a dependent variable depends only on a constant, such is the case for all nine countries for the variable Ds_t^{GER}, then this dependent variable is not listed below.

Australia: $p_t=(p_{t-2}, p_{t-4}, y_{t-1})$, $l_t=(l_{t-1})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{GER}=(l_{t-1}^{Ger})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(p_{t-4})$

Canada: $p_t=(p_{t-2}, l_{t-1}^{GER}, l_{t-2}^{GER})$, $l_t=(l_{t-1})$, $l_t^{US}=(l_{t-1}^{US}, y_{t-1})$, $l_t^{GER}=(l_{t-1}^{GER})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(l_{t-2}^{GER}, y_{t-1})$

Germany: $p_t=(p_{t-3}, p_{t-4}, y_{t-1}, y_{t-2}, y_{t-3})$, $l_t=(l_{t-1}, l_{t-1}^{US}, l_{t-2}^{US})$, $l_t^{US}=(l_{t-1}^{US}, y_{t-4})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(p_{t-1}, y_{t-4})$

Italy: $p_t=(p_{t-1}, l_{t-1}, l_{t-2})$, $l_t=(l_{t-1})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(y_{t-1})$

Japan: $p_t=(p_{t-2}, p_{t-3})$, $l_t=(Ds_{t-1}^{US}, l_{t-1})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{GER}=(l_{t-1}^{GER})$, $y_t=(y_{t-3})$

Netherlands: $p_t=(p_{t-1}, p_{t-3}, p_{t-4})$, $Ds_t^{JAP}=(Ds_{t-1}^{JAP})$, $l_t=(l_{t-1}, l_{t-3}, l_{t-4}, l_{t-1}^{US}, l_{t-2}^{US})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{JAP}=(Ds_{t-1}^{JAP}, l_{t-1}^{JAP})$, $y_t=(p_{t-2}, p_{t-4}, y_{t-1})$

Spain: $p_t=(p_{t-1}, l_{t-2}^{JAP}, l_{t-3}^{JAP})$, $Ds_t^{JAP}=(Ds_{t-1}^{JAP})$, $l_t=(l_{t-1}, l_{t-2})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(y_{t-1})$

UK: $p_t=(p_{t-1}, p_{t-2}, y_{t-1})$, $Ds_t^{US}=(Ds_{t-1}^{US}, y_{t-4})$, $Ds_t^{JAP}=(Ds_{t-1}^{JAP})$, $l_t=(p_{t-1}, l_{t-1}, l_{t-4}^{US}, l_{t-1}^{JAP}, y_{t-4})$, $l_t^{US}=(l_{t-1}^{US})$, $l_t^{GER}=(l_{t-1}^{GER})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(p_{t-3}, y_{t-2})$

United States: $p_t=(p_{t-1}, p_{t-2})$, $l_t=(l_{t-1})$, $l_t^{GER}=(l_{t-1}^{GER})$, $l_t^{JAP}=(l_{t-1}^{JAP})$, $y_t=(p_{t-1}, y_{t-1})$

Appendix 3. Unconstrained Portfolios (Full VAR)

	d_1	d_2^{US}	d_2^{GER}	d_2^{JAP}	d_3	d_4^{US}	d_4^{GER}	d_4^{JAP}
Australia	2.93	0.31	-0.03	-0.56	0.34	-0.61	-0.01	0.80
Canada	-4.70	1.36	-0.38	-2.86	1.36	-1.81	0.51	2.67
Germany	24.62	-0.90		-0.19	-0.09	0.67		-0.02
Italy	-4.40	-0.29		0.53	0.16	-0.11		0.00
Japan	-0.54	0.36	0.09		-0.31	-0.71	0.36	
Netherlands	3.36	-1.07		0.54	-0.75	0.97		-0.41
Spain	-1.42	-0.29		1.48	0.69	0.47		-1.19
UK	6.12	0.65	-0.53	-1.75	-0.42	-0.35	0.84	1.62
USA	6.34		-2.94	0.20	-0.64		2.29	0.02

Note: See Table 1.

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